Line profile formation in a magnetic field

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intuitive understanding

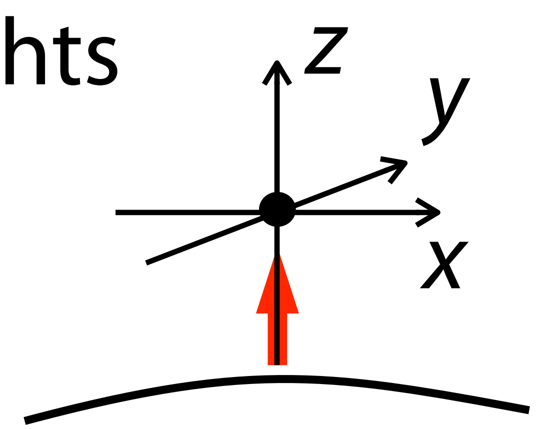
- $J = 0 - 1$ transition is considered
- density matrix for atoms under anisotropic irradiation
- emergence of coherence between magnetic sublevels by rotation of coordinates
- influence of magnetic field on density matrix
- derivation of Stokes parameters from density matrix

density matrix

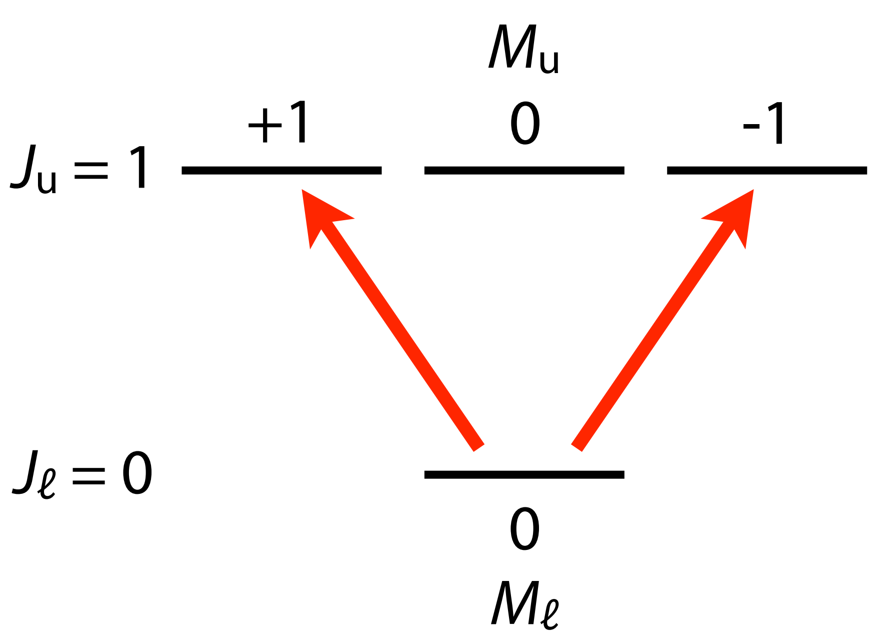
- eigenstates of J_z , $|M\rangle$, are considered and density matrix (operator) is expressed as
$$\rho = \sum_M p_M |M\rangle\langle M|$$
- isotropic case with $J = 1$
$$\rho = \frac{1}{3}\left\{|1\rangle\langle 1| + |0\rangle\langle 0| + |-1\rangle\langle -1|\right\}$$
$$= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- off diagonal components appear when there exists *coherence* between basis states

anisotropic photo-excitation

- unpolarized σ-light can be understood to involve incoherent two circularly polarized lights



- excitation gives rise to anisotropic excited state



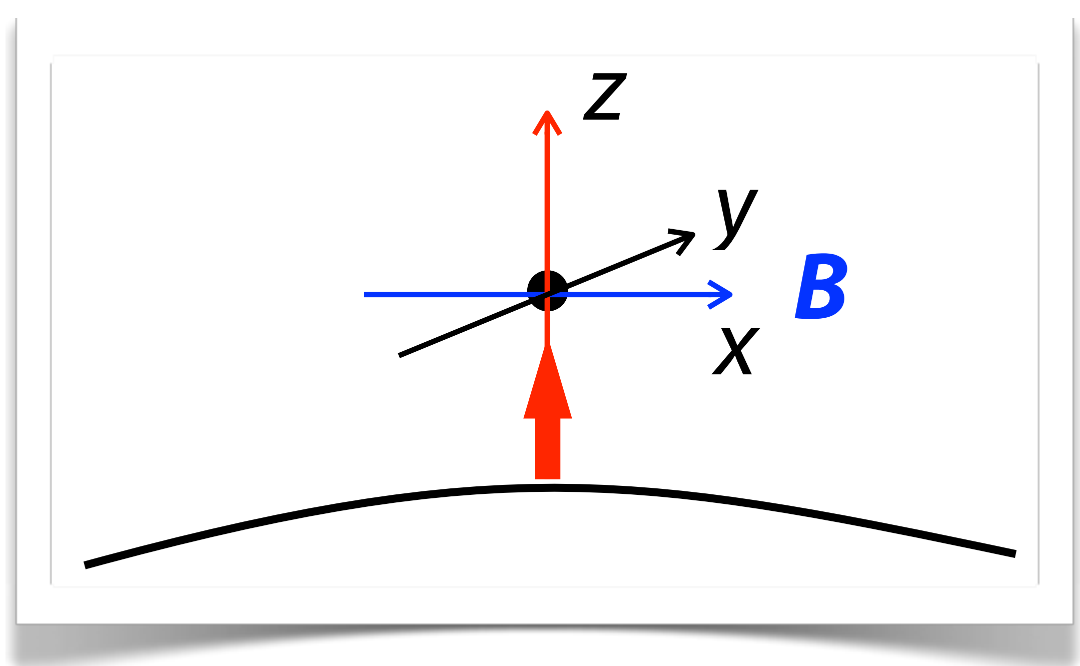
$$\rho = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} M_u = +1 & 0 & -1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ +1 \\ 0 \\ -1 \end{array}$$

- there is no coherence (non-diagonal component) for this moment

- consider a situation with a magnetic field in *x*-axis direction

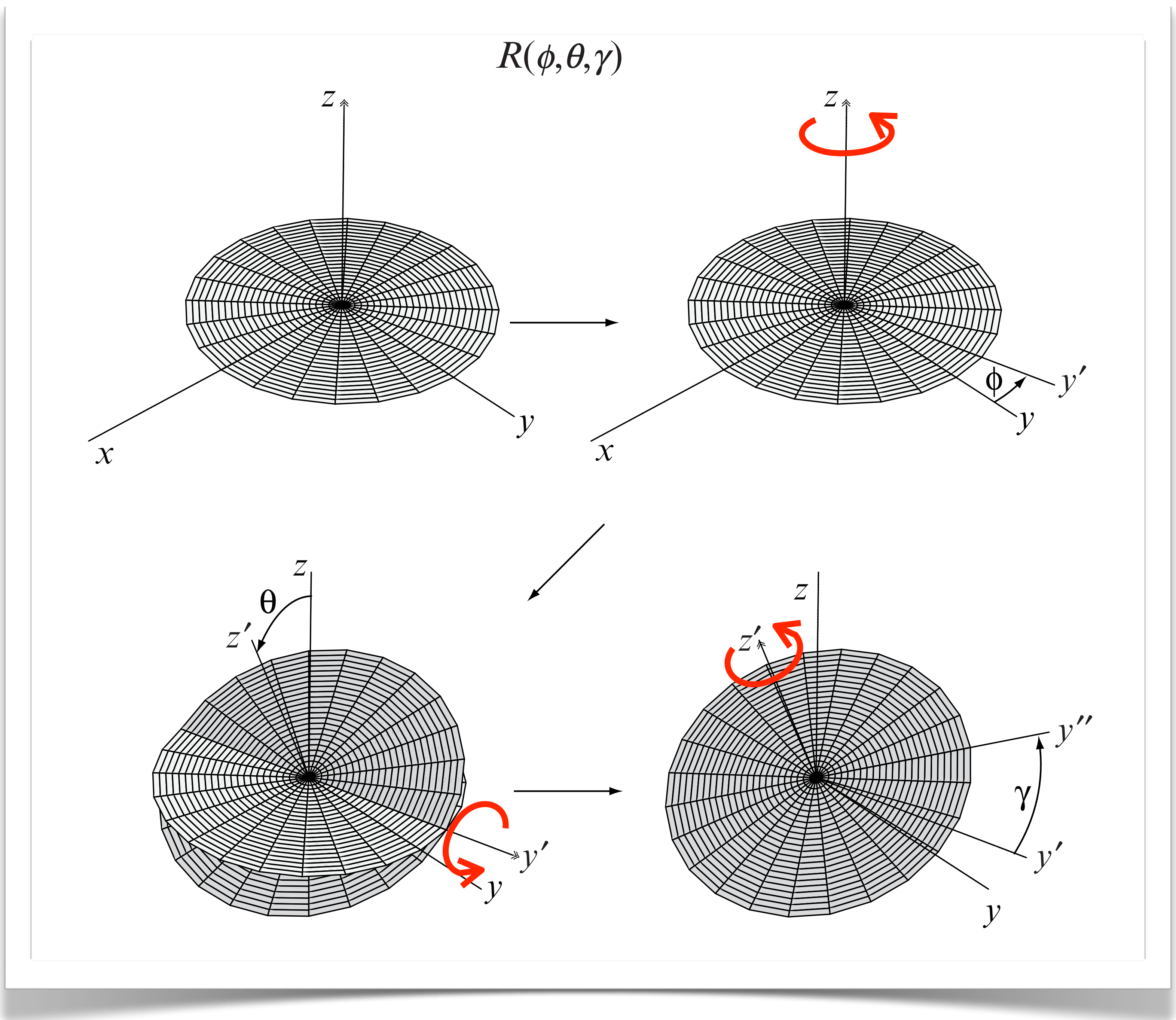
- it would be useful to change the quantization axis from *z*- to *x*-axis

- density matrix is transformed in change of the quantization axis



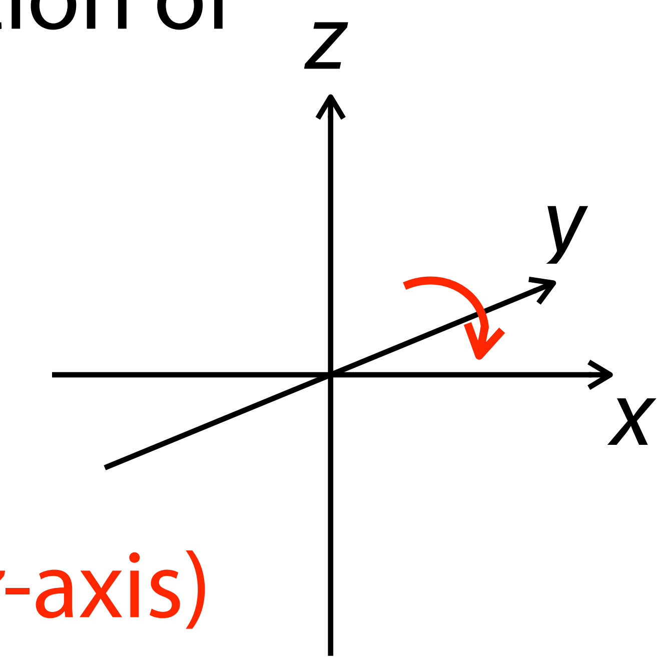
change of quantization axis

- quantization axis change is realized by rotation of coordinates — Euler rotation



rotation operator (matrix)

- coordinates rotation is expressed as action of rotation operator $\mathscr{D}(R)$ to kets or bras



- density matrix elements are formally calculated as

$$\rho_{MN} = \langle M|\rho|N\rangle \text{ (quantization axis} \rightarrow z\text{-axis)}$$

$$\begin{aligned}\rho_{M_xN_x} &= \langle M_x|\rho|N_x\rangle \text{ (quantization axis} \rightarrow x\text{-axis)} \\ &= (\langle M|\mathscr{D}^\dagger(R))\rho(\mathscr{D}(R)|N\rangle) \\ &= \sum_{mn}\langle M|\mathscr{D}^\dagger(R)|m\rangle\langle m|\rho|n\rangle\langle n|\mathscr{D}(R)|N\rangle \\ &= \sum_{mn}\mathscr{D}^{(J)*}_{mM}(R)\mathscr{D}^{(J)}_{nN}(R)\langle m|\rho|n\rangle\end{aligned}$$

closure

rotation operator (matrix)

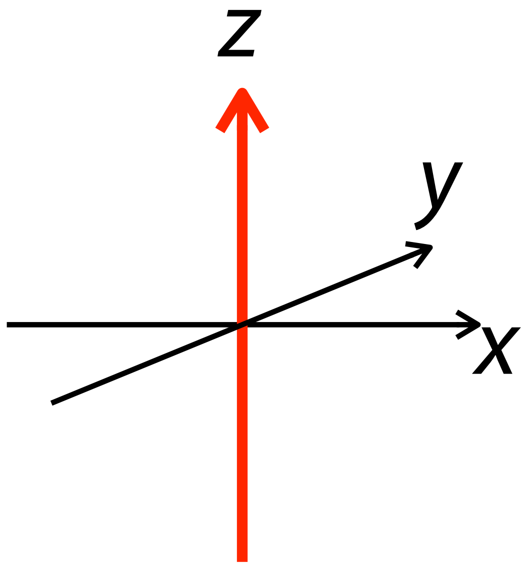
rotation with respect to y-axis

$$\langle M|\mathscr{D}(\alpha,\beta,\gamma)|N\rangle = \mathscr{D}^{(J)}_{MN}(\alpha,\beta,\gamma) = e^{-i(M\alpha+N\gamma)} d^{(J)}_{MN}(\beta)$$

$$d^{(J)}_{MN}(\beta) = \sum_k (-1)^{k-M+N} \frac{\sqrt{(J+M)!(J-M)!(J+N)!(J-N)!}}{(J+M-k)!k!(J-k-N)!(k-M+N)!} \times \left(\cos\frac{\beta}{2}\right)^{2J-2k+M-N} \left(\sin\frac{\beta}{2}\right)^{2k-M+N}$$

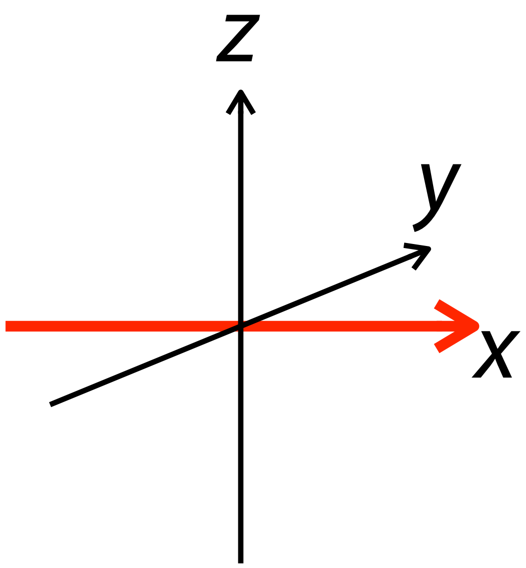
Wigner's formula

$$\rho_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\Downarrow$$

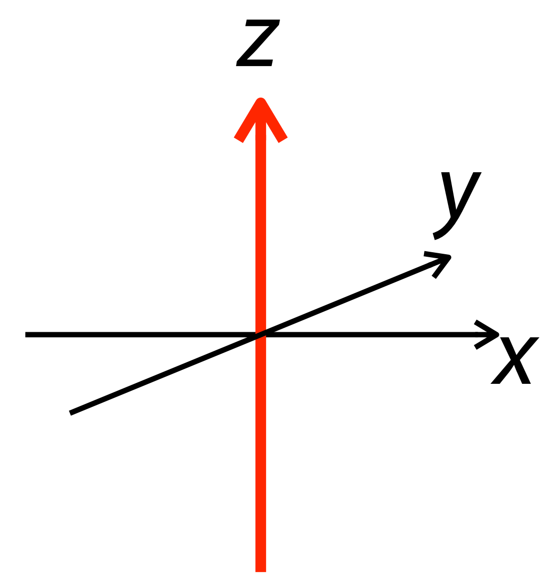
$$\rho_x = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



- coherence emerges between $M = +1$ and $M = -1$ states

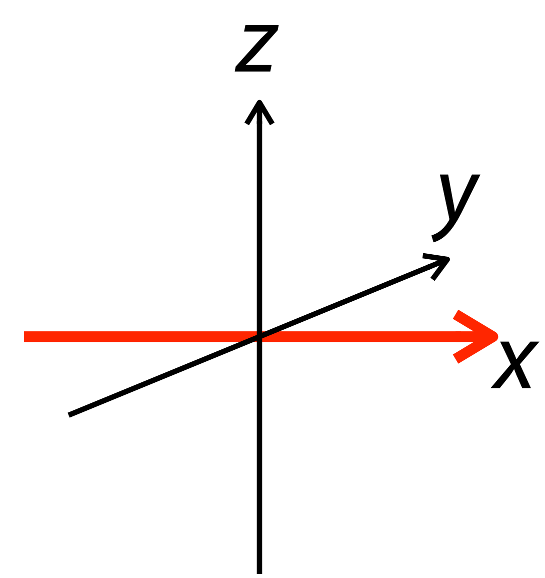
- no coherence appears in isotropic case

$$\rho_z = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\Downarrow$$

$$\rho_x = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



role of magnetic field

- density matrix is derived as solution of equation of motion

$$i\hbar\frac{\partial}{\partial t}\rho_x = [H_\mathrm{F}, \rho_x]$$

- Hamiltonian H_F consists of perturbation due to magnetic field

$$\begin{aligned}\langle M|H_\mathrm{F}|N\rangle &= -\mu_\mathrm{B} g_J B\langle M|J_x|N\rangle \\ &= -\mu_\mathrm{B} g_J B M\delta_{MN} \\ &= -\hbar\omega_0 M\delta_{MN}\end{aligned}$$

- μ_B and g_J are Bohr magneton and Landé g-factor, respectively, and ω_0 corresponds to Larmor angular frequency

- H_F is explicitly written as

$$H_\mathrm{F} = -\hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

equation of motion

$$i\hbar\frac{\partial}{\partial t}\rho_x = [H_\mathrm{F}, \rho_x]$$

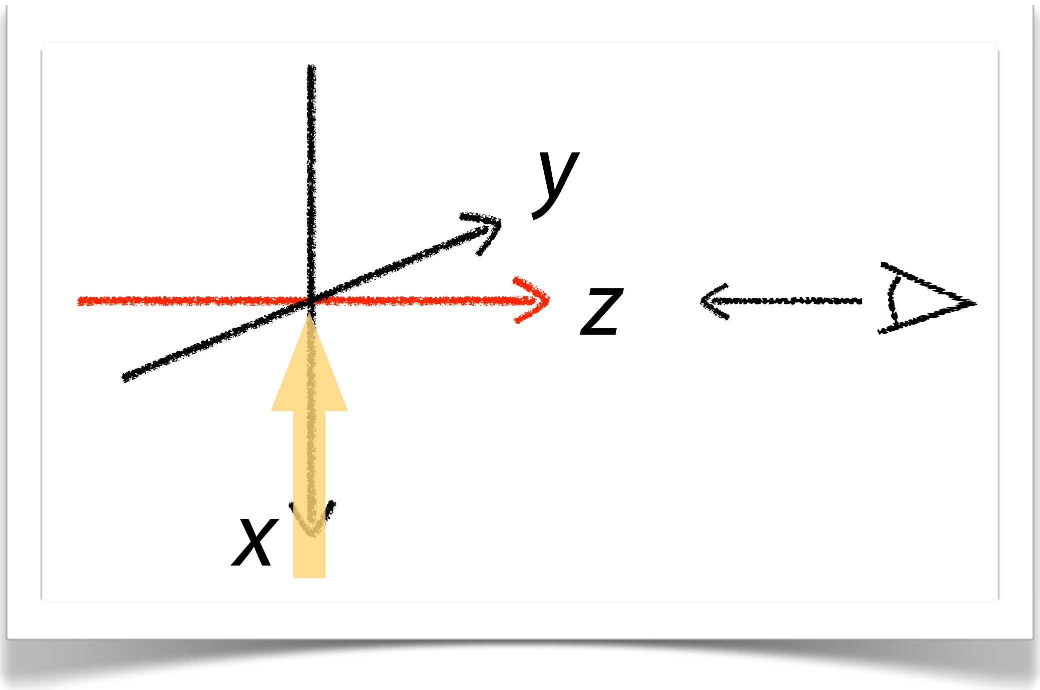
- right hand side of equation is calculated as

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \rho_{1\,1} & \rho_{1\,0} & \rho_{1\,-1} \\ \rho_{0\,1} & \rho_{0\,0} & \rho_{0\,-1} \\ \rho_{-1\,1} & \rho_{-1\,0} & \rho_{-1\,-1} \end{pmatrix} = -\hbar\omega_0 \begin{pmatrix} 0 & \rho_{1\,0} & 2\rho_{1\,-1} \\ -\rho_{0\,1} & 0 & \rho_{0\,-1} \\ -2\rho_{-1\,1} & -\rho_{-1\,0} & 0 \end{pmatrix}$$

- $\rho_x(t)$ is readily obtained with initial condition

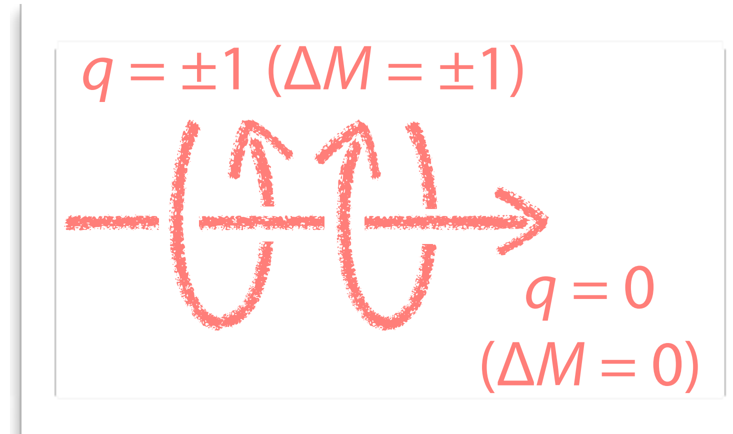
$$\rho_x(t) = \frac{1}{4}\begin{pmatrix} 1 & 0 & e^{2i\omega t} \\ 0 & 2 & 0 \\ e^{-2i\omega t} & 0 & 1 \end{pmatrix}$$

- line intensity is derived from density matrix obtained



line intensity

spherical components



$$I^{q}_{M_\alpha M_\beta} = C_\mathrm{D} \left| \langle \alpha J_\alpha M_\alpha | d_q | \beta J_\beta M_\beta \rangle \right|^2$$

$$I^{q}_{\alpha\beta} = C_\mathrm{D} \sum_{M_\alpha, M_\beta} w_{M_\alpha} |\langle \alpha J_\alpha M_\alpha | d_q | \beta J_\beta M_\beta \rangle|^2$$

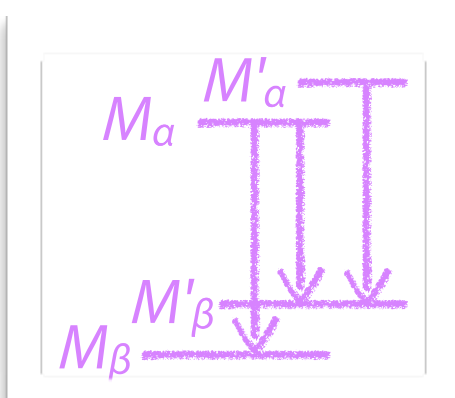
$$= C_\mathrm{D} \sum_{M_\alpha, M_\beta} w_{M_\alpha} \langle \alpha J_\alpha M_\alpha | d_q | \beta J_\beta M_\beta \rangle \langle \beta J_\beta M_\beta | d_q^\dagger | \alpha J_\alpha M_\alpha \rangle$$

$$\left(\sum_{M_\alpha} |\alpha J_\alpha M_\alpha \rangle \langle \alpha J_\alpha M_\alpha | = 1 \right)$$

closure can be inserted anywhere

$$= C_\mathrm{D} \sum_{M_\alpha, M_\beta} w_{M_\alpha} \langle \alpha J_\alpha M_\alpha | \underline{\left(\sum_{M''_\alpha} |\alpha J_\alpha M''_\alpha \rangle \langle \alpha J_\alpha M''_\alpha | \right)} d_q | \beta J_\beta M_\beta \rangle$$

$$\times \langle \beta J_\beta M_\beta | d_q^\dagger \underline{\left(\sum_{M'_\alpha} |\alpha J_\alpha M'_\alpha \rangle \langle \alpha J_\alpha M'_\alpha | \right)} |\alpha J_\alpha M_\alpha \rangle$$



$$
\begin{aligned}
I^q_{\alpha\beta} &= C_\mathrm{D} \sum_{M'_\alpha, M''_\alpha} \sum_{M_\alpha, M_\beta} w_{M_\alpha} \langle \alpha J_\alpha M'_\alpha | \alpha J_\alpha M_\alpha \rangle \langle \alpha J_\alpha M_\alpha | \alpha J_\alpha M''_\alpha \rangle \\
&\quad \times \langle \alpha J_\alpha M''_\alpha | d_q | \beta J_\beta M_\beta \rangle \langle \beta J_\beta M_\beta | d^\dagger_q | \alpha J_\alpha M'_\alpha \rangle \\
&= C_\mathrm{D} \sum_{M'_\alpha, M''_\alpha, M_\beta} \langle \alpha J_\alpha M'_\alpha | \rho_\alpha | \alpha J_\alpha M''_\alpha \rangle \\
&\quad \times \langle \alpha J_\alpha M''_\alpha | d_q | \beta J_\beta M_\beta \rangle \langle \beta J_\beta M_\beta | d^\dagger_q | \alpha J_\alpha M'_\alpha \rangle
\end{aligned}
$$

$$\left(\rho_\alpha = \sum_{M_\alpha} w_{M_\alpha} | \alpha J_\alpha M_\alpha \rangle \langle \alpha J_\alpha M_\alpha | \right)$$

$$= \langle \alpha J_\alpha M'_\alpha | d_q | \beta J_\beta M_\beta \rangle^*$$

$$\left(\begin{aligned} &\langle \alpha J_\alpha M_\alpha | d_q | \beta J_\beta M_\beta \rangle \\ &= (-1)^{J_\beta + M_\alpha + 1} \begin{pmatrix} J_\alpha & J_\beta & 1 \\ -M_\alpha & M_\beta & q \end{pmatrix} \langle \alpha J_\alpha \| \mathbf{d} \| \beta J_\beta \rangle \end{aligned} \right)$$

Wigner-Eckart theorem

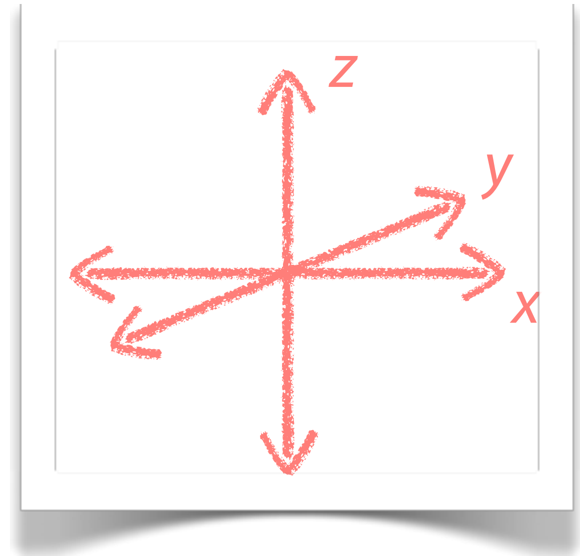
$$
\begin{aligned}
&= C_\mathrm{D} \sum_{M'_\alpha, M''_\alpha, M_\beta} (-1)^{M'_\alpha + M''_\alpha} \langle \alpha J_\alpha M'_\alpha | \rho_\alpha | \alpha J_\alpha M''_\alpha \rangle \\
&\quad \times \begin{pmatrix} J_\alpha & J_\beta & 1 \\ -M''_\alpha & M_\beta & q \end{pmatrix} \begin{pmatrix} J_\alpha & J_\beta & 1 \\ -M'_\alpha & M_\beta & q \end{pmatrix} |\langle \alpha J_\alpha \| \mathbf{d} \| \beta J_\beta \rangle|^2
\end{aligned}
$$

linear polarization components

$d_q \rightarrow d_x$ and d_y

$$d_x = \frac{1}{\sqrt{2}}(d_{-1} - d_1)$$

$$d_y = \frac{\mathrm{i}}{\sqrt{2}}(d_{-1} + d_1)$$



$$I^x_{\alpha\beta} = \frac{C_\mathrm{D}}{2} \sum_{M'_\alpha, M''_\alpha, M_\beta} \langle \alpha J_\alpha M'_\alpha | \rho_\alpha | \alpha J_\alpha M''_\alpha \rangle$$

$$\times \langle \alpha J_\alpha M''_\alpha | \underline{d_{-1} - d_1} | \beta J_\beta M_\beta \rangle \langle \beta J_\beta M_\beta | \underline{d^\dagger_{-1} - d^\dagger_1} | \alpha J_\alpha M'_\alpha \rangle$$

$$I^y_{\alpha\beta} = \frac{C_\mathrm{D}}{2} \sum_{M'_\alpha, M''_\alpha, M_\beta} \langle \alpha J_\alpha M'_\alpha | \rho_\alpha | \alpha J_\alpha M''_\alpha \rangle$$

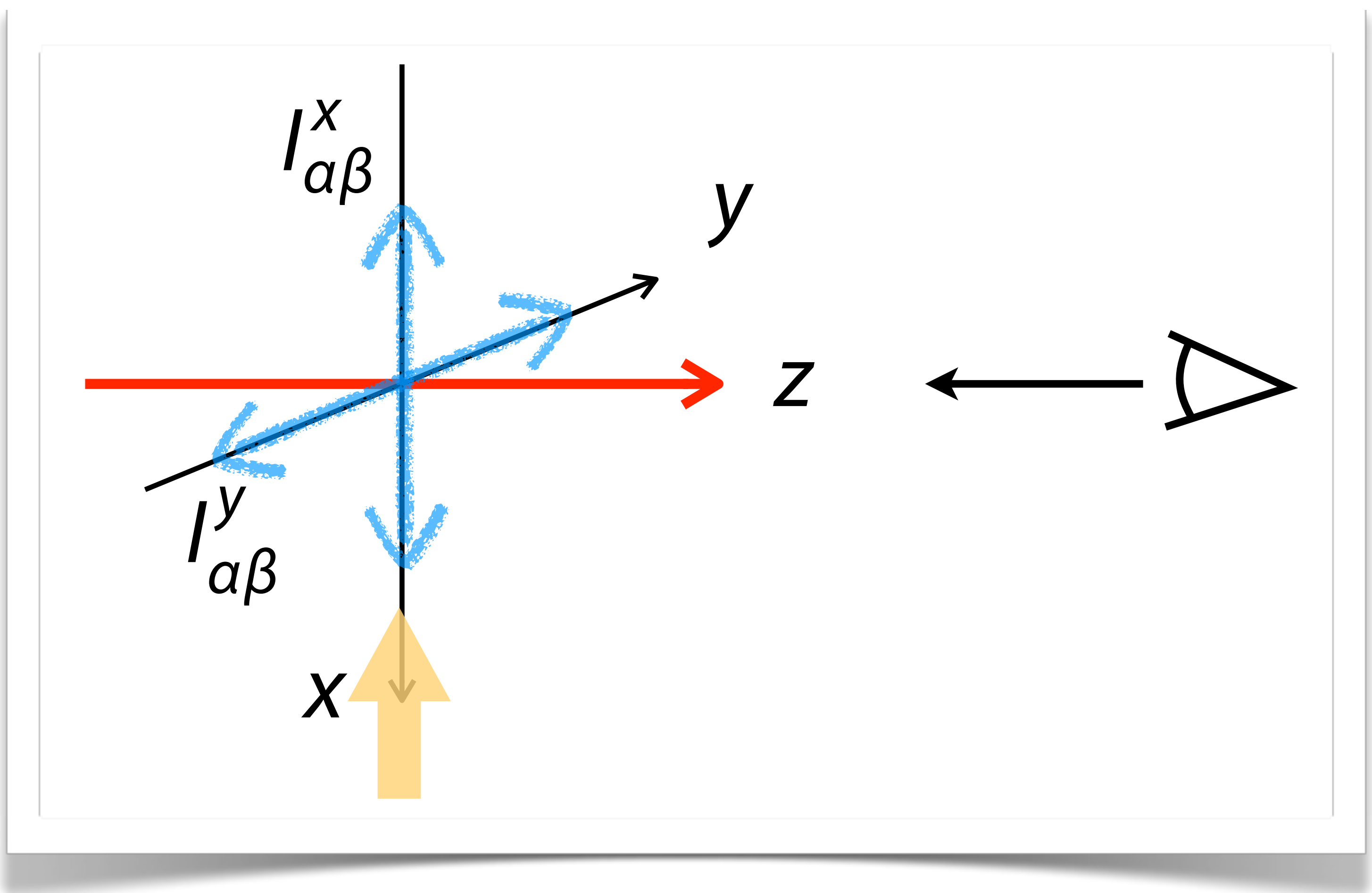
$$\times \langle \alpha J_\alpha M''_\alpha | \underline{d_{-1} + d_1} | \beta J_\beta M_\beta \rangle \langle \beta J_\beta M_\beta | \underline{d^\dagger_{-1} + d^\dagger_1} | \alpha J_\alpha M'_\alpha \rangle$$

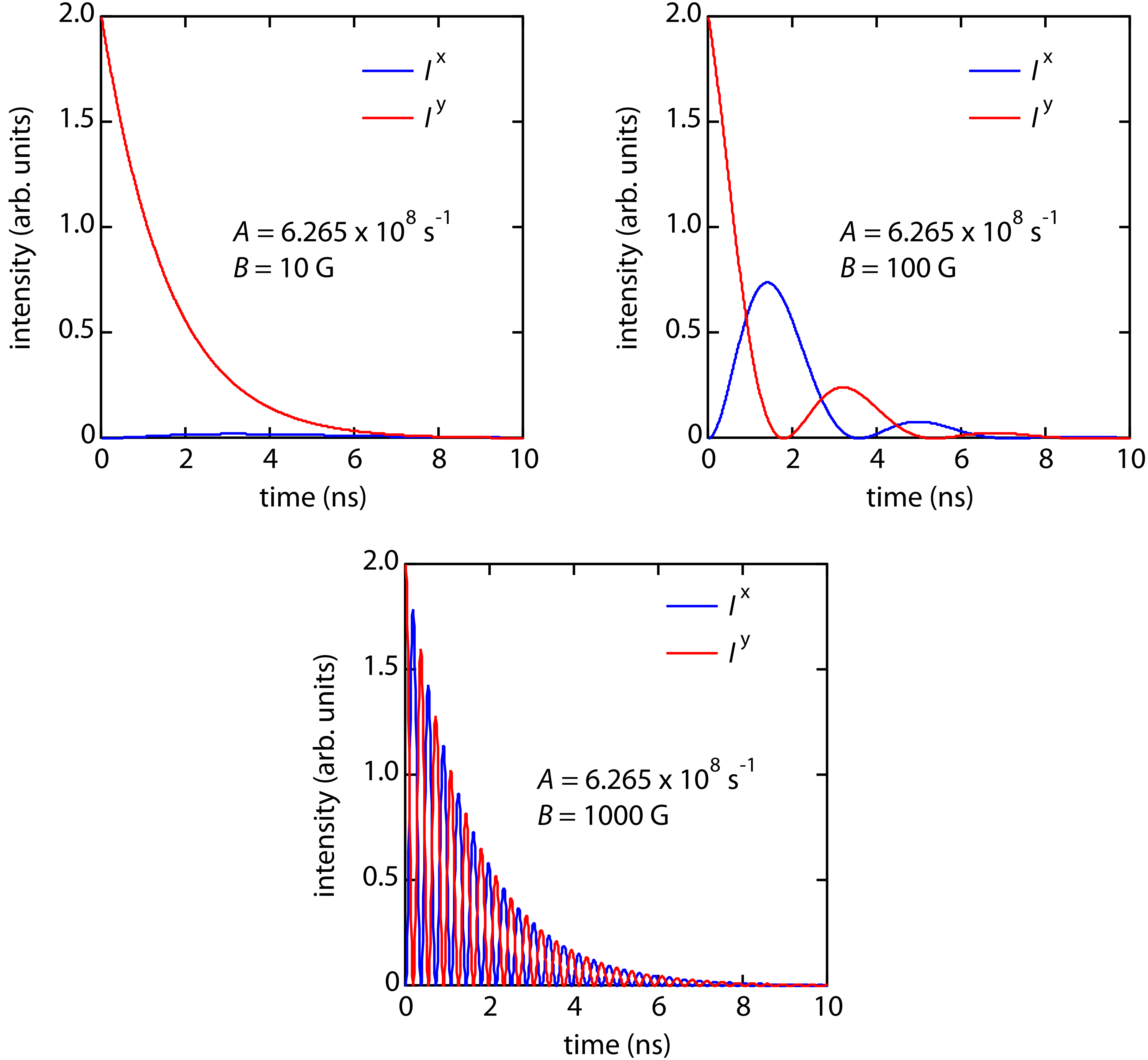
$$I^x_{\alpha\beta} = \frac{C_D}{12}(1 - \cos 2\omega t)|\langle \alpha 1 || \mathbf{d} || \beta 0 \rangle|^2 \times \exp(-A_{\alpha\beta} t)$$

$$I^y_{\alpha\beta} = \frac{C_D}{12}(1 + \cos 2\omega t)|\langle \alpha 1 || \mathbf{d} || \beta 0 \rangle|^2 \times \exp(-A_{\alpha\beta} t)$$

$$\omega = \frac{e}{2m} g_J B$$

spontaneous decay





I^x
I^y
$A = 6.265 \times 10^8\ s^{-1}$
$B = 10$ G
intensity (arb. units)
time (ns)
I^x
I^y
$A = 6.265 \times 10^8\ s^{-1}$
$B = 100$ G
intensity (arb. units)
time (ns)
I^x
I^y
$A = 6.265 \times 10^8\ s^{-1}$
$B = 1000$ G
intensity (arb. units)
time (ns)

more systematic method

- density matrix and Stokes parameters are derived in accordance with "*Polarization in Spectral Lines*" by E. Landi Degl'Innocenti and M. Landolfi
- correspondence to the intuitive method of the results is considered

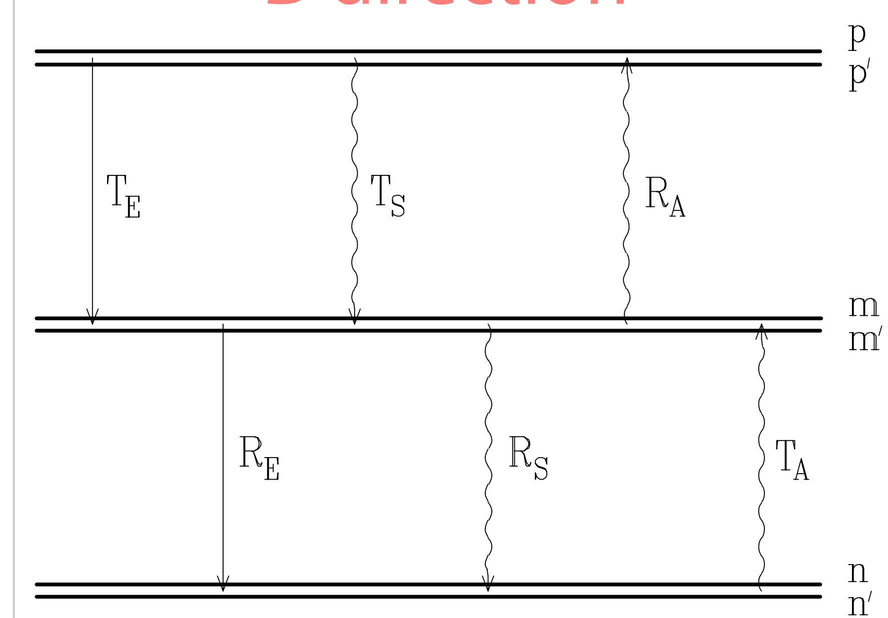
equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \frac{2\pi}{\mathrm{i}h}[H,\rho]$$

- Hamiltonian can involve atomic processes in addition to magnetic field (QED is required)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\alpha J}(M,M') &= -2\pi\mathrm{i}\,\nu_{\mathrm{L}}\,g_{\alpha J}\,(M-M')\,\rho_{\alpha J}(M,M') \\
&+ \sum_{\alpha_\ell J_\ell}\sum_{M_\ell M'_\ell} \rho_{\alpha_\ell J_\ell}(M_\ell,M'_\ell)\,T_{\mathrm{A}}(\alpha J M M',\alpha_\ell J_\ell M_\ell M'_\ell) \\
&+ \sum_{\alpha_u J_u}\sum_{M_u M'_u} \rho_{\alpha_u J_u}(M_u,M'_u)\Big[T_{\mathrm{E}}(\alpha J M M',\alpha_u J_u M_u M'_u) \\
&\qquad + T_{\mathrm{S}}(\alpha J M M',\alpha_u J_u M_u M'_u)\Big] \\
&- \sum_{M''}\Big\{\rho_{\alpha J}(M,M'')\Big[R_{\mathrm{A}}(\alpha J M' M'') + R_{\mathrm{E}}(\alpha J M'' M') \\
&\qquad + R_{\mathrm{S}}(\alpha J M'' M')\Big] \\
&\quad + \rho_{\alpha J}(M'',M')\Big[R_{\mathrm{A}}(\alpha J M'' M) + R_{\mathrm{E}}(\alpha J M M'') \\
&\qquad + R_{\mathrm{S}}(\alpha J M M'')\Big]\Big\}
\end{aligned}$$

quantization axis in ***B*** direction



standard representation

spherical tensors

- spherical representation of density matrix is obtained from standard matrix as

$$\rho^K_Q(\alpha J, \alpha J) = \rho^K_Q(\alpha J)$$

$$= \sum_{MM'} (-1)^{J-M} \sqrt{2K+1} \begin{pmatrix} J & J & K \\ M & -M' & -Q \end{pmatrix} \rho_{\alpha J}(M, M')$$

  where $K = 0, 1, ..., 2J$ and $Q = -K, ..., K$

- it is understood as change of *basis* to express matrices: e.g., for $J = 1/2$

$$\rho(M,N) : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho^K_Q : \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

advantages

- standard representation requires two rotation matrices in rotation of coordinates,

$$\Big[\rho_{\alpha J}(M, M')\Big]_{\mathrm{new}} = \sum_{NN'} \mathcal{D}^{J}_{NM}(R)^{*}\, \mathcal{D}^{J}_{N'M'}(R)\, \Big[\rho_{\alpha J}(N, N')\Big]_{\mathrm{old}}$$

  while spherical representation needs just one rotation matrix

$$\Big[\rho^{K}_{Q}(\alpha J, \alpha' J')\Big]_{\mathrm{new}} = \sum_{Q'} \Big[\rho^{K}_{Q'}(\alpha J, \alpha' J')\Big]_{\mathrm{old}}\, \mathcal{D}^{K}_{Q'Q}(R)^{*}$$

- many components vanish when there exists some symmetry

spherical representation

- multiplying both sides in equation of motion by

$$(-1)^{J-M}\sqrt{2K+1}\begin{pmatrix} J & J & K \\ M & -M' & -Q \end{pmatrix}$$

and carrying out summation over *M* and *M′* give

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho^K_Q(\alpha J) = -2\pi\mathrm{i}\,\nu_\mathrm{L}\;g_{\alpha J}\;Q\;\rho^K_Q(\alpha J)$$

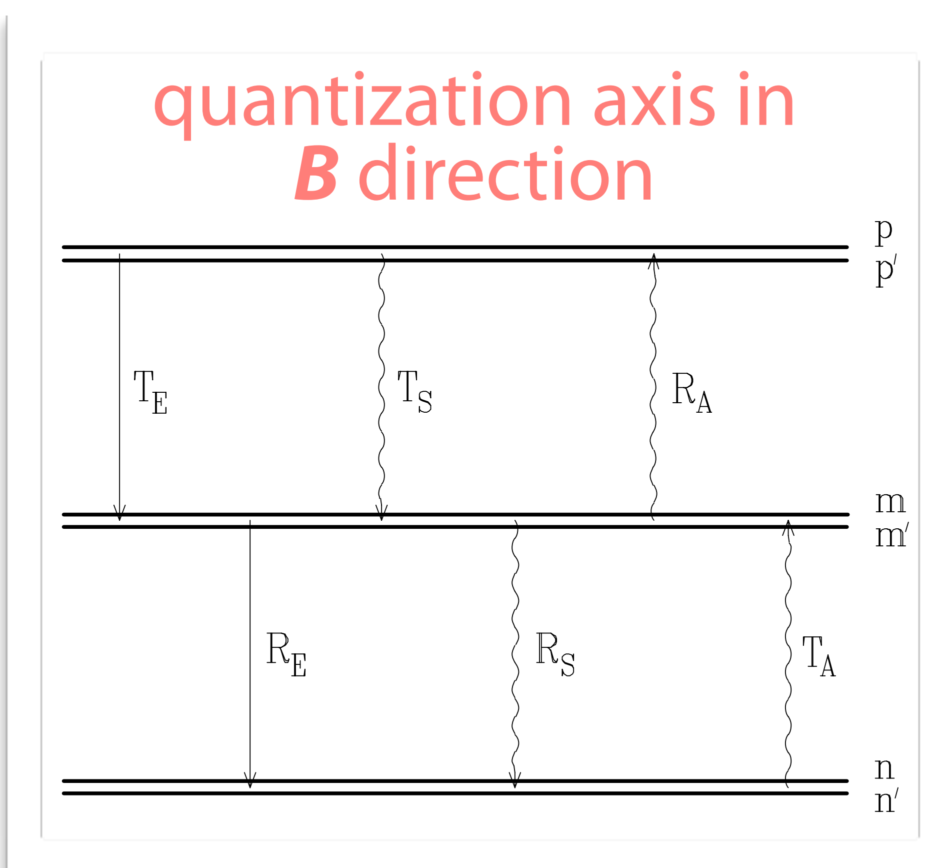
$$+\sum_{\alpha_\ell J_\ell}\sum_{K_\ell Q_\ell}\rho^{K_\ell}_{Q_\ell}(\alpha_\ell J_\ell)\;\mathbb{T}_\mathrm{A}(\alpha JKQ,\alpha_\ell J_\ell K_\ell Q_\ell)+$$

$$+\sum_{\alpha_u J_u}\sum_{K_u Q_u}\rho^{K_u}_{Q_u}(\alpha_u J_u)\Big[\mathbb{T}_\mathrm{E}(\alpha JKQ,\alpha_u J_u K_u Q_u)$$

$$+\,\mathbb{T}_\mathrm{S}(\alpha JKQ,\alpha_u J_u K_u Q_u)\Big]$$

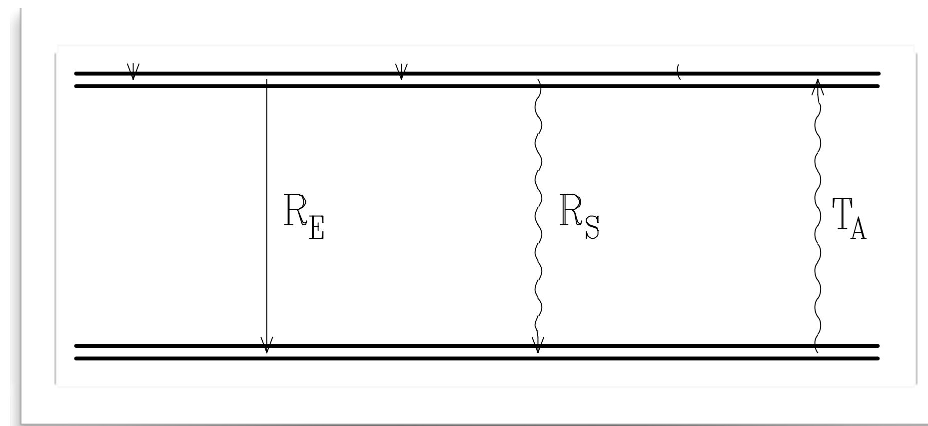
$$-\sum_{K'Q'}\rho^{K'}_{Q'}(\alpha J)\Big[\mathbb{R}_\mathrm{A}(\alpha JKQK'Q')+\mathbb{R}_\mathrm{E}(\alpha JKQK'Q')$$

$$+\,\mathbb{R}_\mathrm{S}(\alpha JKQK'Q')\Big]$$



two-level atom

- upper level



$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho^K_Q(\alpha_u J_u) = -2\pi\mathrm{i}\,\nu_\mathrm{L}\; g_{\alpha_u J_u}\; Q\;\rho^K_Q(\alpha_u J_u)$$

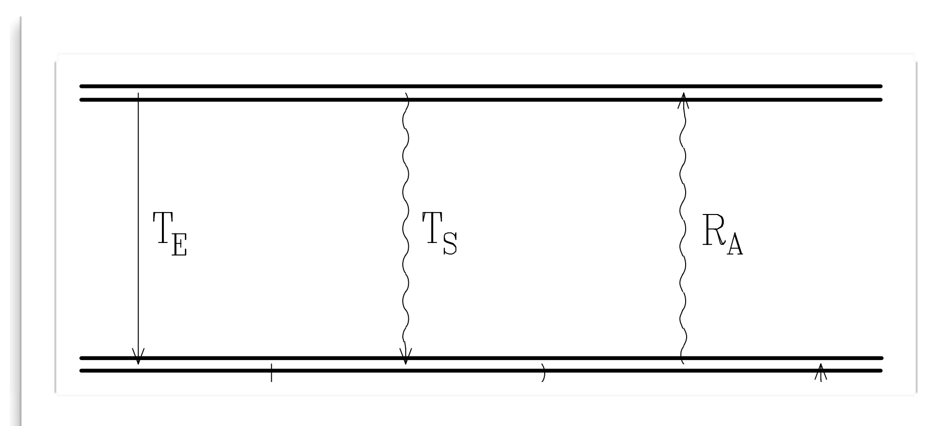
$$+\sum_{K'Q'} \mathbb{T}_\mathrm{A}(\alpha_u J_u K Q, \alpha_\ell J_\ell K'Q')\;\rho^{K'}_{Q'}(\alpha_\ell J_\ell)$$

$$-\sum_{K'Q'}\Big[\mathbb{R}_\mathrm{E}(\alpha_u J_u K Q K'Q') + \mathbb{R}_\mathrm{S}(\alpha_u J_u K Q K'Q')\Big]\;\rho^{K'}_{Q'}(\alpha_u J_u)$$

$$\delta_{KK'}\,\delta_{QQ'}\sum_{\alpha_\ell J_\ell} A(\alpha J \to \alpha_\ell J_\ell)$$

ignored

- lower level



$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho^K_Q(\alpha_\ell J_\ell) = -2\pi\mathrm{i}\,\nu_\mathrm{L}\; g_{\alpha_\ell J_\ell}\; Q\;\rho^K_Q(\alpha_\ell J_\ell)$$

$$+\sum_{K'Q'}\Big[\mathbb{T}_\mathrm{E}(\alpha_\ell J_\ell K Q, \alpha_u J_u K'Q') + \mathbb{T}_\mathrm{S}(\alpha_\ell J_\ell K Q, \alpha_u J_u K'Q')\Big]\;\rho^{K'}_{Q'}(\alpha_u J_u)$$

$$-\sum_{K'Q'} \mathbb{R}_\mathrm{A}(\alpha_\ell J_\ell K Q K'Q')\;\rho^{K'}_{Q'}(\alpha_\ell J_\ell)$$

- when stationary and lower level is unpolarized

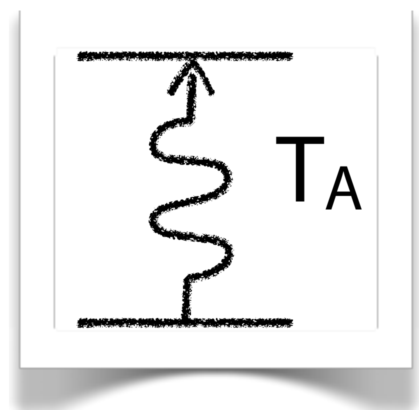
only this term remains

$$\rho^K_Q(\alpha_u J_u) = \frac{\mathbb{T}_\mathrm{A}(\alpha_u J_u K Q, \alpha_\ell J_\ell\, 0\, 0)}{2\pi \mathrm{i}\nu_\mathrm{L} g_{\alpha_u J_u} Q + A(\alpha_u J_u \to \alpha_\ell J_\ell)} \times \rho^0_0(\alpha_\ell J_\ell)$$

$$\mathbb{T}_\mathrm{A}(\alpha J K Q, \alpha_\ell J_\ell \cancelto{0}{K_\ell} \cancelto{0}{Q_\ell}) = (2J_\ell + 1)\, B(\alpha_\ell J_\ell \to \alpha J)$$

$$\times \sum_{K_\mathrm{r} Q_\mathrm{r}} \sqrt{3(2K+1)(2\cancelto{0}{K_\ell}+1)(2K_\mathrm{r}+1)}$$

$$\times (-1)^{\cancelto{0}{K_\ell} + \cancelto{0}{Q_\ell}} \begin{Bmatrix} J & J_\ell & 1 \\ J & J_\ell & 1 \\ K & \cancelto{0}{K_\ell} & K_\mathrm{r} \end{Bmatrix} \underbrace{\begin{pmatrix} K & \cancelto{0}{K_\ell} & K_\mathrm{r} \\ -Q & \cancelto{0}{Q_\ell} & -Q_\mathrm{r} \end{pmatrix}}_{K_\mathrm{r} = K,\ Q_\mathrm{r} = -Q} \underbrace{J^{K_\mathrm{r}}_{Q_\mathrm{r}}(\nu_{\alpha J, \alpha_\ell J_\ell})}_{\text{radiation field tensor}}$$



$$\mathbb{T}_{\mathrm{A}}(\alpha_u J_u K Q, \alpha_\ell J_\ell\, 0\, 0)$$

$$= (2J_\ell + 1) B(\alpha_\ell J_\ell \to \alpha_u J_u) \times \sqrt{3(2K+1)^2}$$

$$\times \begin{Bmatrix} J_u & J_\ell & 1 \\ J_u & J_\ell & 1 \\ K & 0 & K \end{Bmatrix} \begin{pmatrix} K & 0 & K \\ -Q & 0 & Q \end{pmatrix} J^K_{-Q}(\nu_{\alpha_u J_u, \alpha_\ell J_\ell})$$

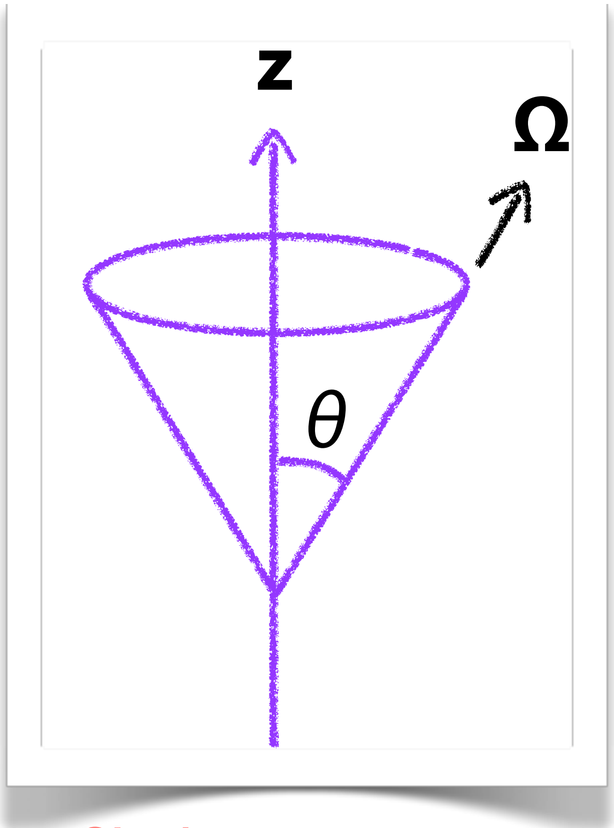
$$= \sqrt{3(2J_\ell + 1)} B(\alpha_\ell J_\ell \to \alpha_u J_u)$$

$$\times (-1)^{1+J_u+J_\ell+Q} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix} J^K_{-Q}(\nu_{\alpha_u J_u, \alpha_\ell J_\ell})$$

Stokes parameters

$$J^K_Q(\nu) = \oint \frac{\mathrm{d}\Omega}{4\pi} \mathcal{I}^K_Q(\nu, \vec{\Omega}) = \oint \frac{\mathrm{d}\Omega}{4\pi} \sum_{i=0}^{3} \mathcal{T}^K_Q(i, \vec{\Omega})\, S_i(\nu, \vec{\Omega})$$

geometrical factors



for unpolarized radiation having *z*-axis symmetry

$$J^0_0(\nu) = \oint \frac{\mathrm{d}\Omega}{4\pi} I(\nu, \theta)$$

$$J^2_0(\nu) = \frac{1}{2\sqrt{2}} \oint \frac{\mathrm{d}\Omega}{4\pi} (3\cos^2\theta - 1)\, I(\nu, \theta)$$ (= 0 for isotropic filed)

$$\rho_Q^K(\alpha_u J_u) = \sqrt{\frac{2J_\ell + 1}{2J_u + 1}} \, \frac{B(\alpha_\ell J_\ell \to \alpha_u J_u)}{A(\alpha_u J_u \to \alpha_\ell J_\ell) + 2\pi \mathrm{i}\, \nu_\mathrm{L} \; g_{\alpha_u J_u} \; Q}$$

$$\times \; w_{J_u J_\ell}^{(K)} \, (-1)^Q \; J_{-Q}^K(\nu_0) \;\; \rho_0^0(\alpha_\ell J_\ell)$$

$$w_{J_u J_\ell}^{(K)} = (-1)^{1+J_\ell+J_u} \sqrt{3(2J_u+1)} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix}$$

$$\boxed{\rho_Q^K(\alpha_u J_u) = \frac{1}{1 + \mathrm{i}\, Q H_u} \left[\rho_Q^K(\alpha_u J_u) \right]_{B=0}}$$

essence of Hanle effect

$$H_u = \frac{2\pi \nu_\mathrm{L} \; g_{\alpha_u J_u}}{A(\alpha_u J_u \to \alpha_\ell J_\ell)}$$

$$\rho^K_Q(\alpha_u J_u) = \frac{1}{1 + \mathrm{i}\, Q H_u} \left[\rho^K_Q(\alpha_u J_u)\right]_{B=0}$$

- if QH_u is expressed to be $\tan(\alpha)$, $\rho^K_Q(\alpha_u J_u)$ is rewritten as

$$\rho^K_Q(\alpha_u J_u) = \mathrm{e}^{-\mathrm{i}\alpha} \cos\alpha \left[\rho^K_Q(\alpha_u J_u)\right]_{B=0}$$

- effect of magnetic field is to reduce by factor of

$$\cos\alpha = \sqrt{\frac{1}{1 + Q^2 H_u^2}}$$

and to dephase by

$$\tan^{-1} Q H_u$$

e.g.

$$\rho_x(t) = \frac{1}{4} \begin{pmatrix} 1 & 0 & e^{2i\omega t} \\ 0 & 2 & 0 \\ e^{-2i\omega t} & 0 & 1 \end{pmatrix}$$

Stokes parameters

frequency-integrated

$$\tilde{\varepsilon}_i(\vec{\Omega}) = \int_{\Delta\nu} \varepsilon_i(\nu, \vec{\Omega})\, \mathrm{d}\nu$$

$$\begin{aligned}
\tilde{\varepsilon}_i(\vec{\Omega}) &= \frac{h^2\nu^4}{2\pi c^2}\, \mathcal{N}\, (2J_u+1)\, B(\alpha_u J_u \to \alpha_\ell J_\ell) \\
&\quad \times \sum_{KQ} \sqrt{3}\, (-1)^{1+J_\ell+J_u} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix} \mathcal{T}^K_Q(i,\vec{\Omega})\, \rho^K_Q(\alpha_u J_u) \\
&= \frac{h\nu}{4\pi}\, \mathcal{N} \sqrt{2J_u+1}\, A(\alpha_u J_u \to \alpha_\ell J_\ell) \sum_{KQ} w^{(K)}_{J_u J_\ell}\, \mathcal{T}^K_Q(i,\vec{\Omega})\, \rho^K_Q(\alpha_u J_u) \\
&= k_{\mathrm{L}}^{\mathrm{A}} \oint \frac{\mathrm{d}\Omega'}{4\pi} \sum_{j=0}^{3} P_{ij}(\vec{\Omega},\vec{\Omega}';\vec{B})\, I_j(\nu_0,\vec{\Omega}')
\end{aligned}$$

$$\left(P_{ij}(\vec{\Omega},\vec{\Omega}';\vec{B}) = \sum_{KQ} W_K(J_\ell, J_u)\, (-1)^Q\, \mathcal{T}^K_Q(i,\vec{\Omega})\, \mathcal{T}^K_{-Q}(j,\vec{\Omega}') \frac{1}{1+\mathrm{i}\,Q H_u} \right)$$

- for $J_\ell = 1$ and $J_u = 2$

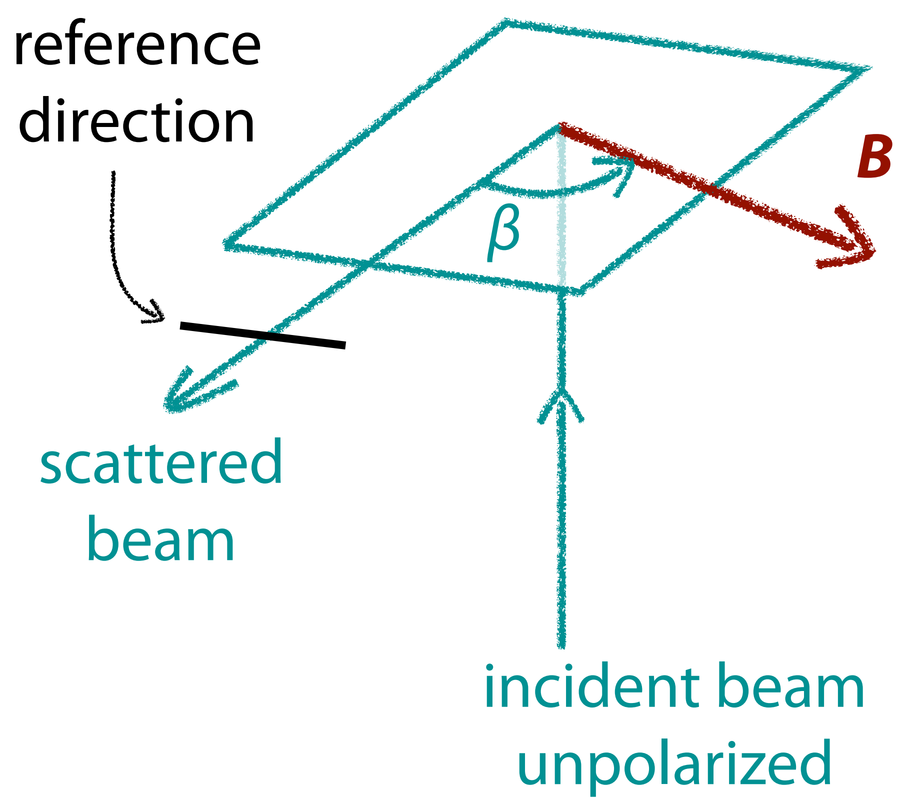
$$\tilde{p}_Q \equiv \frac{\tilde{\varepsilon}_Q(\vec{\Omega})}{\tilde{\varepsilon}_I(\vec{\Omega})} = \frac{3\,W_2\left[\sin^2\beta + (1+\cos^2\beta)\cos^2\alpha_2\right]}{8 + W_2\,(1 - 3\cos^2\beta - 3\sin^2\beta\cos^2\alpha_2)}$$

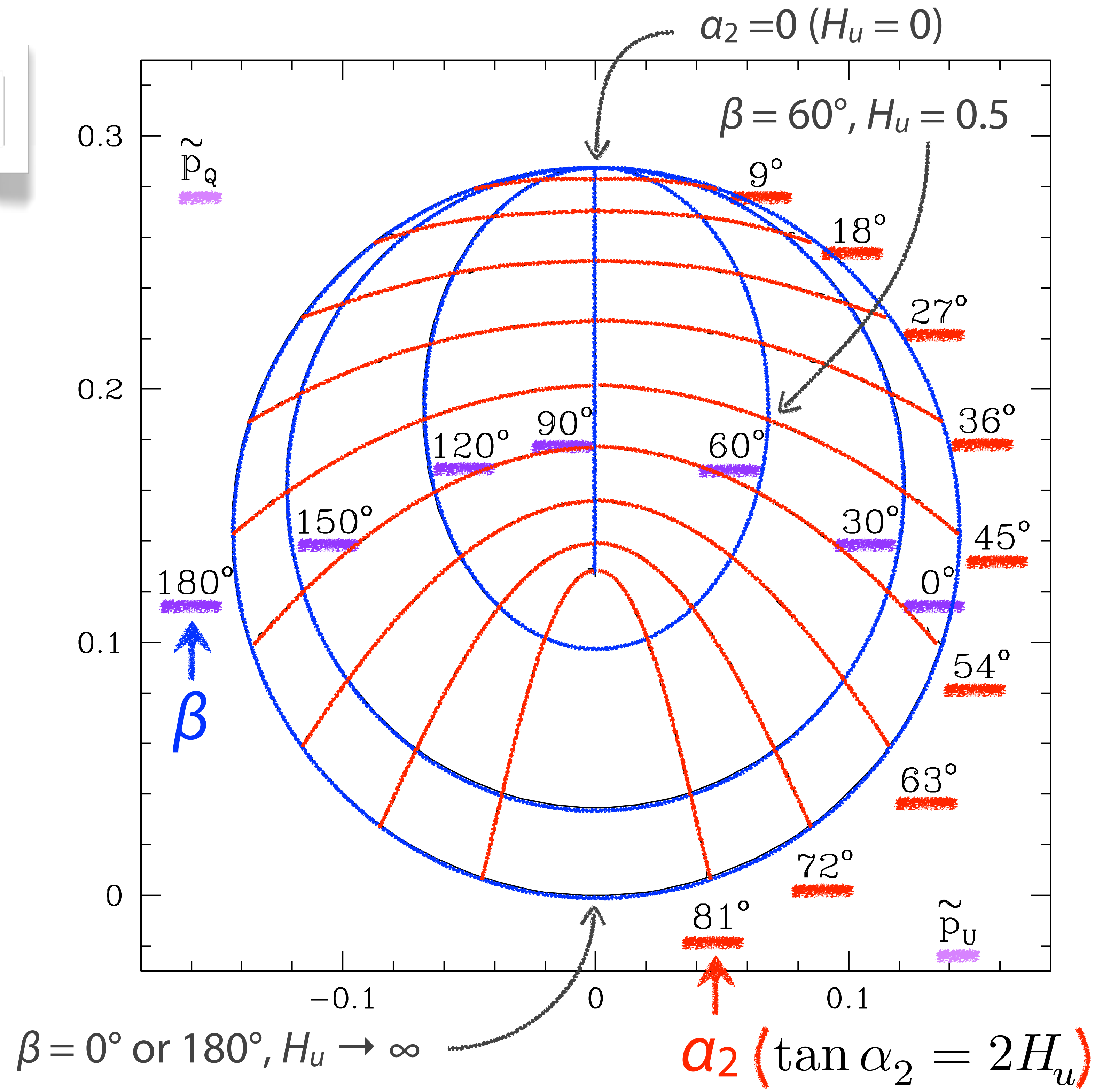
$$\tilde{p}_U \equiv \frac{\tilde{\varepsilon}_U(\vec{\Omega})}{\tilde{\varepsilon}_I(\vec{\Omega})} = \frac{6\,W_2\,\cos\beta\,\sin\alpha_2\cos\alpha_2}{8 + W_2\,(1 - 3\cos^2\beta - 3\sin^2\beta\cos^2\alpha_2)}$$

where $\tan\alpha_2 = 2H_u$

Hanle diagram

$J_\ell = 1, J_u = 2$





detailed line profile

$$\varepsilon_i(\nu,\vec{\Omega}) = \frac{2h\nu^3}{c^2}\ \eta_i^{\rm S}(\nu,\vec{\Omega})$$

$$\begin{aligned}
\eta_i^{\rm S}(\nu,\vec{\Omega}) = \frac{h\nu}{4\pi}\mathcal{N} \sum_{\alpha_\ell J_\ell} \sum_{\alpha_u J_u} & (2J_u+1)\, B(\alpha_u J_u \rightarrow \alpha_\ell J_\ell) \\
\times \sum_{K Q K_u Q_u} & \sqrt{3(2K+1)(2K_u+1)} \\
\times \sum_{M_u M_u' M_\ell q q'} & (-1)^{1+J_u-M_u+q'} \begin{pmatrix} J_u & J_\ell & 1 \\ -M_u & M_\ell & -q \end{pmatrix} \begin{pmatrix} J_u & J_\ell & 1 \\ -M_u' & M_\ell & -q' \end{pmatrix} \\
& \times \begin{pmatrix} 1 & 1 & K \\ q & -q' & -Q \end{pmatrix} \begin{pmatrix} J_u & J_u & K_u \\ M_u' & -M_u & -Q_u \end{pmatrix} \\
& \times \mathrm{Re}\Big[\mathcal{T}_Q^K(i,\vec{\Omega})\ \rho_{Q_u}^{K_u}(\alpha_u J_u)\ \varPhi(\nu_{\alpha_u J_u M_u,\, \alpha_\ell J_\ell M_\ell} - \nu) \Big]
\end{aligned}$$

$$\begin{aligned}
\varPhi(\nu_{ab}-\nu) &= \phi(\nu_{ab}-\nu) + \mathrm{i}\,\psi(\nu_{ab}-\nu) \\
&= \frac{1}{\pi}\frac{\varGamma_{ab}}{\varGamma_{ab}^2 + (\nu_{ab}+\varDelta_{ab}-\nu)^2} + \frac{\mathrm{i}}{\pi}\frac{\nu_{ab}+\varDelta_{ab}-\nu}{\varGamma_{ab}^2 + (\nu_{ab}+\varDelta_{ab}-\nu)^2}\ ,
\end{aligned}$$

where

$$\varGamma_{ab} = \frac{\gamma_{ab}}{4\pi} = \frac{\gamma_a+\gamma_b}{4\pi}\ , \qquad \varDelta_{ab} = \varDelta_a - \varDelta_b\ ,$$

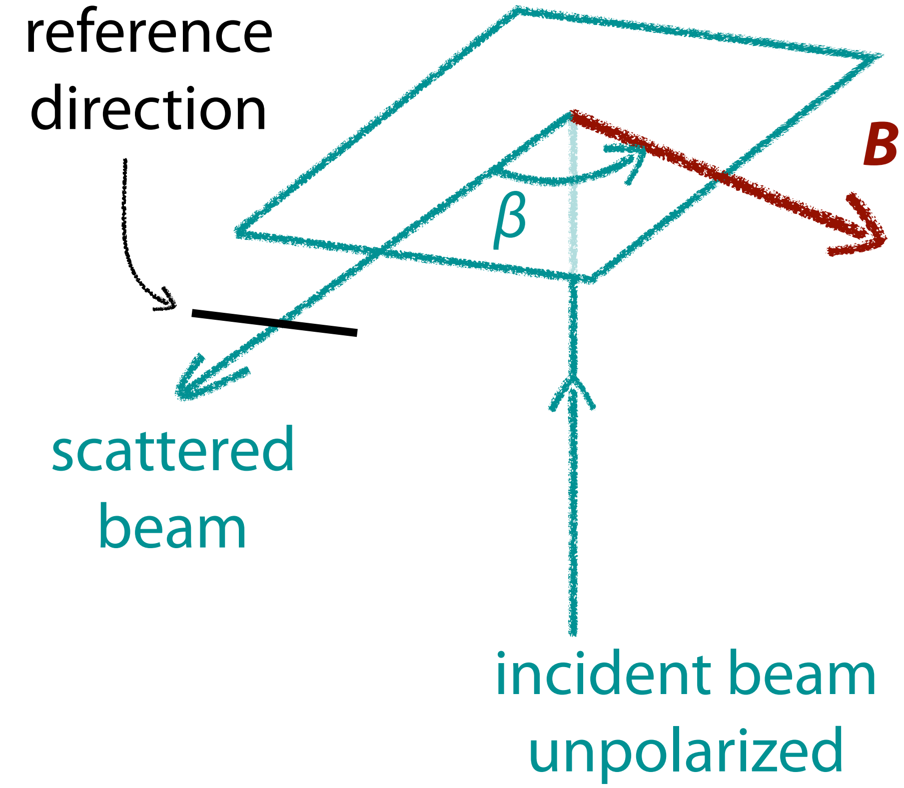
$$\varepsilon_i(\nu,\vec{\Omega}) = \frac{h\nu}{4\pi}\,\mathcal{N}\,\sqrt{2J_u+1}\;A(\alpha_u J_u \to \alpha_\ell J_\ell)$$
$$\times \sum_{KK'Q} \mathcal{T}_Q^{K'}(i,\vec{\Omega})\;\rho_Q^K(\alpha_u J_u)\;\underline{\Phi_Q^{KK'}(J_\ell,J_u;\nu)}$$

generalized profile

$$\underline{\Phi_Q^{KK'}(J_\ell,J_u;\nu)} = \sqrt{3(2J_u+1)(2K+1)(2K'+1)}$$
$$\times \sum_{M_u M_u' M_\ell q q'} (-1)^{1+J_u-M_u+q'} \begin{pmatrix} J_u & J_\ell & 1 \\ -M_u & M_\ell & -q \end{pmatrix} \begin{pmatrix} J_u & J_\ell & 1 \\ -M_u' & M_\ell & -q' \end{pmatrix}$$
$$\times \begin{pmatrix} J_u & J_u & K \\ M_u' & -M_u & -Q \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ q & -q' & -Q \end{pmatrix}$$
$$\times \frac{1}{2}\Big[\Phi(\nu_{\alpha_u J_u M_u,\,\alpha_\ell J_\ell M_\ell}-\nu) + \Phi(\nu_{\alpha_u J_u M_u',\,\alpha_\ell J_\ell M_\ell}-\nu)^*\Big]\,.$$

- substitution of $\rho_Q^K(\alpha_u J_u)$ gives

$$\varepsilon_i(\nu,\vec{\Omega}) = k_{\mathrm{L}}^{\mathrm{A}} \sum_{KK'Q} \Phi_Q^{KK'}(J_\ell,J_u;\nu)$$
$$\times \oint \frac{\mathrm{d}\Omega'}{4\pi} \sum_{j=0}^{3} w_{J_u J_\ell}^{(K)}\;(-1)^Q\,\mathcal{T}_Q^{K'}(i,\vec{\Omega})\,\mathcal{T}_{-Q}^{K}(j,\vec{\Omega}')\,\frac{1}{1+\mathrm{i}\,QH_u}\,I_j(\nu_0,\vec{\Omega}')$$



- $J_\ell = 0$, $J_u = 1$ and $\beta = 0°$

$$\varepsilon_0(\nu,\vec{\Omega}) = \frac{3}{8}\,k_{\mathrm{L}}^{\mathrm{A}}\,\Delta\Omega'\,I'\Big[\phi_{-1}+\phi_1\Big]$$

$$\varepsilon_1(\nu,\vec{\Omega}) = \frac{3}{8}\,k_{\mathrm{L}}^{\mathrm{A}}\,\Delta\Omega'\,I'\left[\frac{1}{1+4H_u^2}\,(\phi_{-1}+\phi_1)-\frac{2H_u}{1+4H_u^2}\,(\psi_{-1}-\psi_1)\right]$$

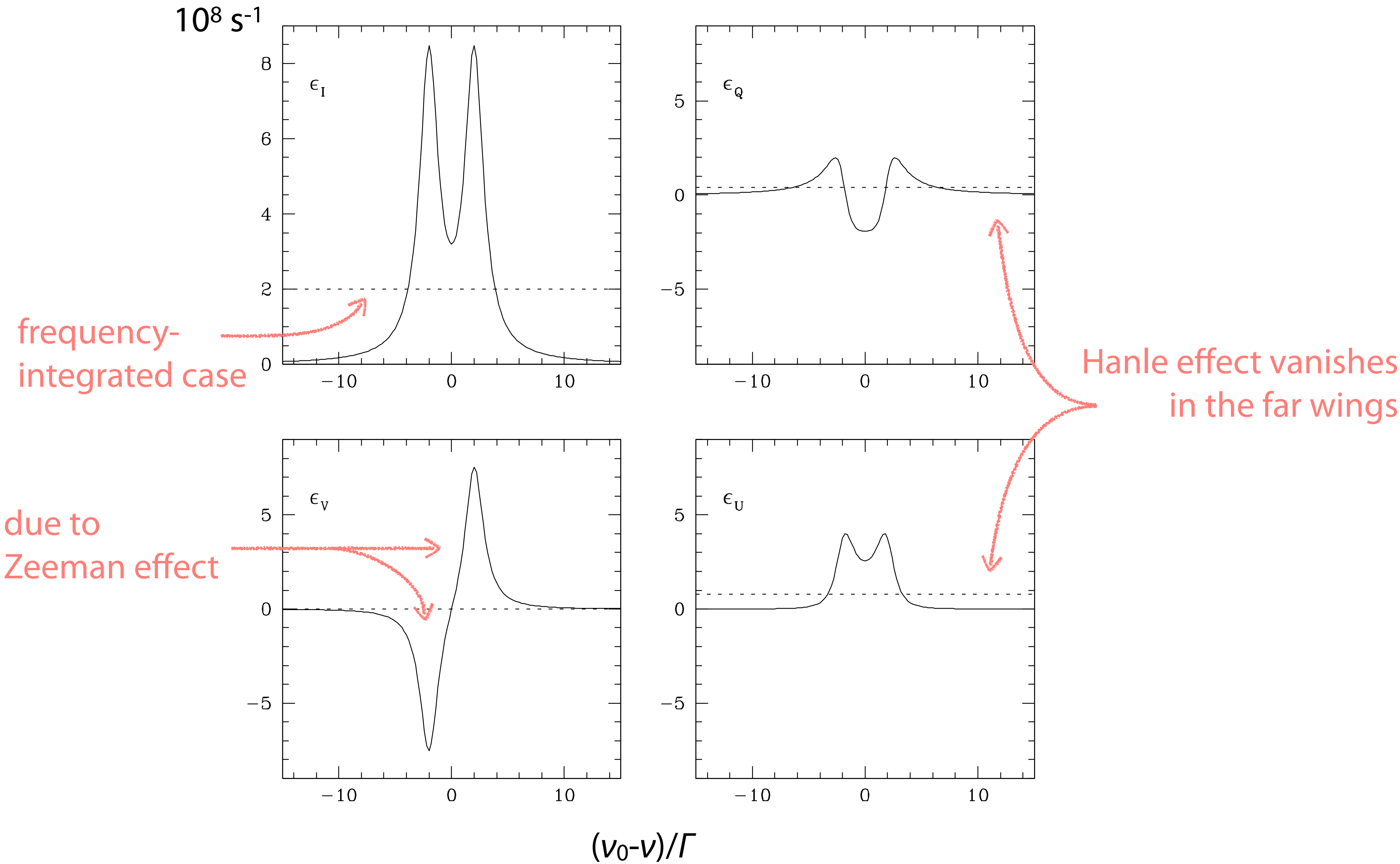
$$\varepsilon_2(\nu,\vec{\Omega}) = \frac{3}{8}\,k_{\mathrm{L}}^{\mathrm{A}}\,\Delta\Omega'\,I'\left[\frac{2H_u}{1+4H_u^2}\,(\phi_{-1}+\phi_1)+\frac{1}{1+4H_u^2}\,(\psi_{-1}-\psi_1)\right]$$

$$\varepsilon_3(\nu,\vec{\Omega}) = -\frac{3}{8}\,k_{\mathrm{L}}^{\mathrm{A}}\,\Delta\Omega'\,I'\Big[\phi_{-1}-\phi_1\Big]$$

$$\varPhi_q = \phi_q + \mathrm{i}\,\psi_q = \varPhi(\nu_{\alpha_u 1\,-q,\;\alpha_\ell 00}-\nu)$$

$A = 5 \times 10^7$ s^{-1}

$H_u = 1$ ($\Gamma = 3.98 \times 10^6$ s^{-1}, $B = 5.69$ G)



next problems

- radiation field tensors should be derived as solution of radiation transport equation
- collisional transitions should be taken into account: excitation, depolarization, . . .
- . . .