

# **Radiation Transfer with Scattering Process**

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- ▶ Radiation Transfer Equation with Scattering Process
- ▶ Line Profile Formation
- ▶ Scattering Processes
  - ✓ Complete “Frequency” Re-Distribution (CRD)
  - ✓ Partial “Frequency” Re-Distribution (PRD)

# Radiative Transfer Equations with Scattering Process

“Classical” Radiative Transfer Equation

$$\frac{1}{c} \frac{\partial \mathbf{I}_\nu}{\partial t} + \mathbf{n} \cdot \nabla \mathbf{I}_\nu = \chi_\nu (\mathbf{S}_\nu - \mathbf{I}_\nu) \longrightarrow \mathbf{n} \cdot \nabla \mathbf{I}_\nu = \chi_\nu (\mathbf{S}_\nu - \mathbf{I}_\nu)$$

$\mathbf{I}_\nu (\mathbf{r}, t; \mathbf{n})$  : specific intensity

$\chi_\nu (\mathbf{r}, t; \mathbf{n}) = \kappa_\nu + \sigma_\nu$  : extinction coefficient

Source Function

$$\mathbf{S}_\nu = \frac{\varepsilon_\nu}{4\pi\chi_\nu} + \alpha_\nu \oint \phi(\mathbf{n}; \mathbf{n}') \mathbf{I}_\nu(\mathbf{n}') d\Omega'$$

$\varepsilon_\nu (\mathbf{r}, t; \mathbf{n})$  : emissivity

$\alpha_\nu = \frac{\sigma_\nu}{\chi_\nu}$  : scattering albedo       $\phi(\mathbf{n}; \mathbf{n}')$  : phase function

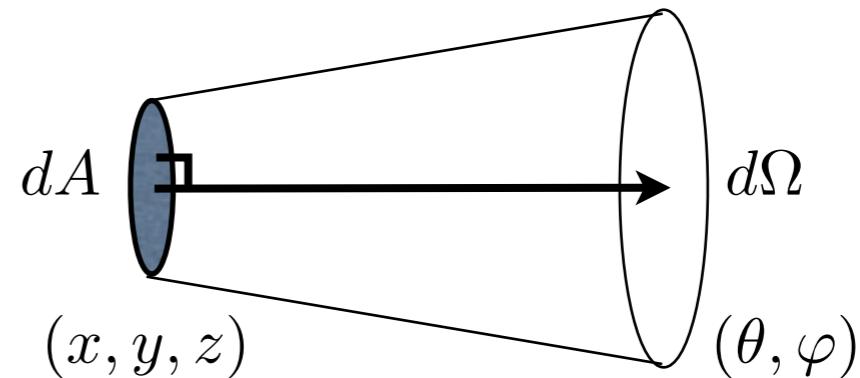
# Derivation of RT Equations #0

## Description of the Radiation Field

- Intensity

$$I(x, y, z, t; \theta, \varphi, \nu)$$

$$dE = I dt dA d\Omega d\nu$$



- Photon distribution function:  $f_R$

$$I(x, t; \mathbf{n}, \nu) = ch\nu \frac{h^3 \nu^2}{c^3} f_R(x, t; \mathbf{n}, \nu)$$

$f_R$ : 6-dimensional phase space

# Derivation of RT Equations #1

## Relation to Boltzmann Transport Equation

- Boltzmann Transport Equation

$$\frac{\partial f_R}{\partial t} + \mathbf{v} \cdot \frac{\partial f_R}{\partial \mathbf{x}} + \mathbf{F} \cdot \frac{\partial f_R}{\partial \mathbf{p}} = \left( \frac{Df_R}{Dt} \right)_{\text{coll}}$$

- Radiative Transfer Equation

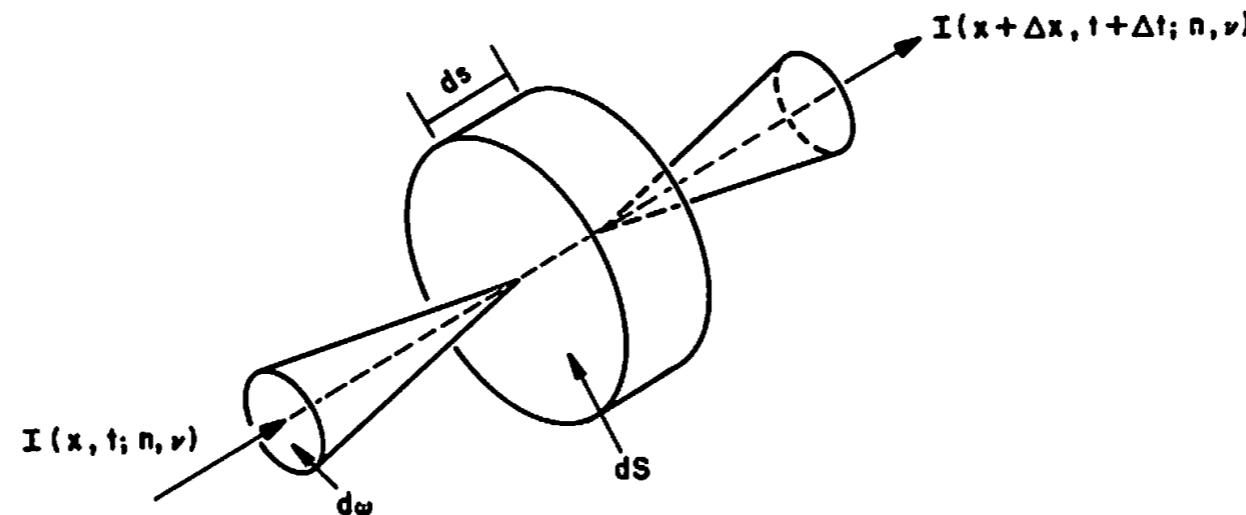
$$\begin{aligned} -\frac{1}{c} \frac{\partial I}{\partial t} + \mathbf{n} \cdot \nabla I &= \chi (S - I) \\ &= \eta - \chi I \end{aligned}$$

$\eta(\mathbf{x}, t; \mathbf{n}, \nu)$  : emissivity

$\chi(\mathbf{x}, t; \mathbf{n}, \nu)$  : opacity per unit length

# Derivation of RT Equations #2

**Classical, macroscopic, and phenomenological derivation**



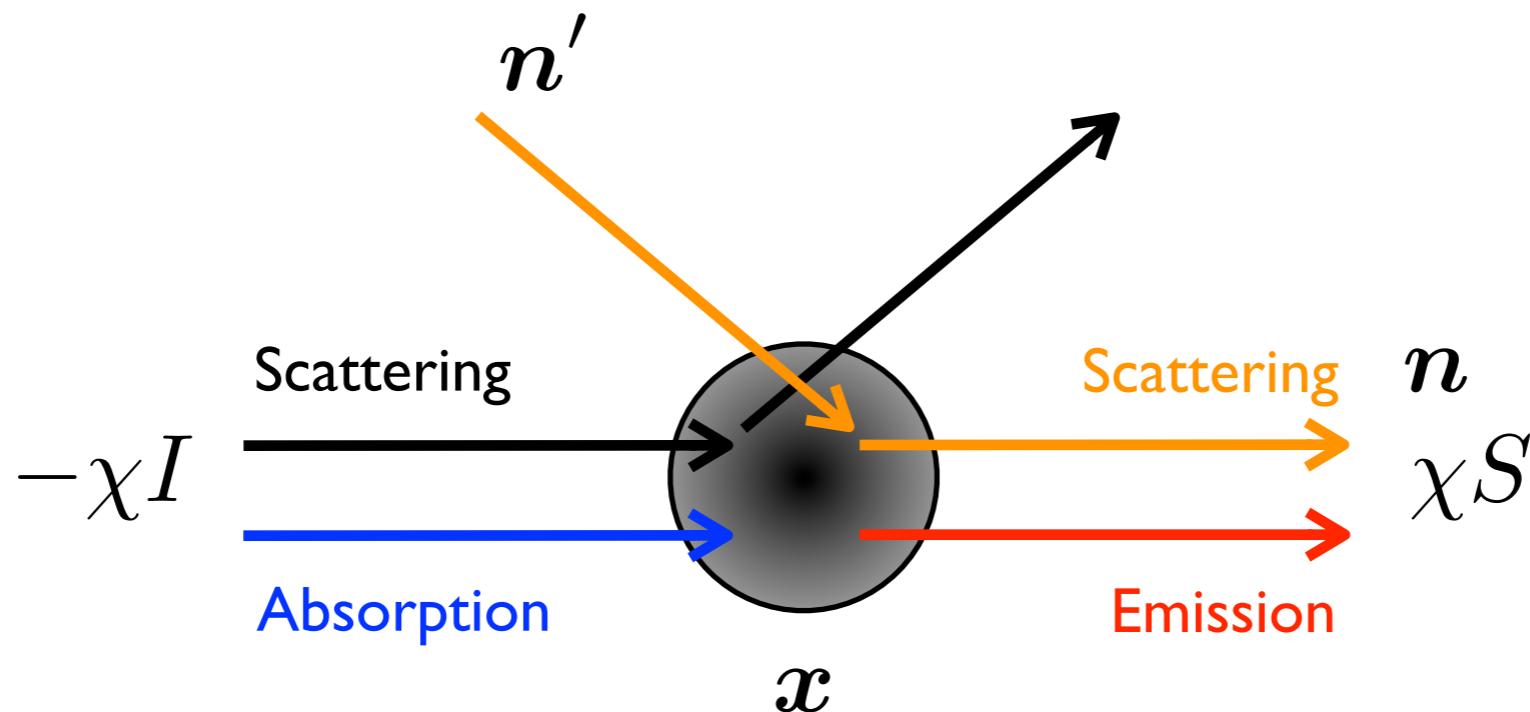
**Fig. 76.1** Pencil of radiation passing through a material element.

$$\begin{aligned} & [I(x + \Delta x, t + \Delta t; \mathbf{n}, \nu) - I(x, t; \mathbf{n}, \nu)] dS dt d\omega d\nu \\ &= \left[ \frac{1}{c} \frac{\partial I(x, t; \mathbf{n}, \nu)}{\partial t} + \frac{\partial I(x, t; \mathbf{n}, \nu)}{\partial s} \right] ds dS dt d\omega d\nu \\ &= [\eta(x, t; \mathbf{n}, \nu) - \chi(x, t; \mathbf{n}, \nu) I(x, t; \mathbf{n}, \nu)] ds dS dt d\omega d\nu \end{aligned}$$

$$\boxed{\frac{1}{c} \frac{\partial I(x, t; \mathbf{n}, \nu)}{\partial t} + \frac{\partial I(x, t; \mathbf{n}, \nu)}{\partial s} = \eta(x, t; \mathbf{n}, \nu) - \chi(x, t; \mathbf{n}, \nu) I(x, t; \mathbf{n}, \nu)}$$

# Derivation of RT Equations #3

## Schematics of RT Equation with Scattering



$$\frac{1}{c} \frac{\partial I}{\partial t} + \mathbf{n} \cdot \nabla I = \left[ \chi_{\text{abs}} B + \chi_{\text{sca}} \int I(\mathbf{n}') \phi(\mathbf{n}, \mathbf{n}') d\Omega \right] - (\chi_{\text{abs}} + \chi_{\text{sca}}) I$$

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mathbf{n} \cdot \nabla I = \chi (S - I)$$

# Radiative Transfer Equations

## Opacity and Level populations

### Opacity

$$\kappa(\nu_{ij}) = \left( \frac{\pi e^2}{m_0 c} \right) n_i f_{ij} \Phi_{ij}(\nu) \quad \Phi_{ij}(\nu) : \text{the spectral line profile function}$$
$$\sigma(\nu) = \frac{\pi e^2}{m_0 c} \frac{2\nu^2 \left( \frac{\nu}{\nu_0} \right)^2 \frac{\gamma}{2\pi^2}}{(\nu^2 - \nu_0^2)^2 + \nu^2 \left( \frac{\gamma}{2\pi} \right)^2} f_s \rightarrow \frac{\pi e^2}{m_0 c} \frac{\gamma}{4\pi} \left[ \frac{1}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2} \right] f_s$$

### Rate Equations (Spontaneous + Induced + Collisional rates)

$$\frac{dn_i}{dt} = -n_i \sum_{j \neq i} (A_{ij} + B_{ij} U_{\nu_{ij}} + C_{ij}) + \sum_{j \neq i} n_j (A_{ji} + B_{ji} U_{\nu_{ij}} + C_{ji})$$

$A_{ij}, B_{ij}$  : radiative processes

$C_{ij}$  : collisional processes

$U_{\nu_{ij}}$  radiation energy density in the range between  $h\nu_{ij} = \varepsilon_i - \varepsilon_j$

# Line Profile Formation

## Overview

- Gaussian Profile

$$\Phi(\nu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\nu - \nu_0)^2}{2\sigma^2}\right]$$

- Lorentz Profile

$$\Phi(\nu) = \frac{\Delta\nu_L/2\pi}{(\nu - \nu_0)^2 + (\Delta\nu_L/2)^2}$$

- The Voigt Profile

$$\Phi(\nu) = \int_{-\infty}^{+\infty} \frac{\Delta\nu_L \exp(-\Delta\nu^2/\Delta\nu_D^2) d(\Delta\nu)}{2\pi\sqrt{\pi}\Delta\nu_D \left[(\nu - \nu_0 - \Delta\nu)^2 + (\Delta\nu_L/2)^2\right]}$$

# Gaussian & Lorentz Profile

## Doppler, Resonance, Collision, Natural Broadening

$$\Phi(\nu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\nu - \nu_0)^2}{2\sigma^2}\right]$$

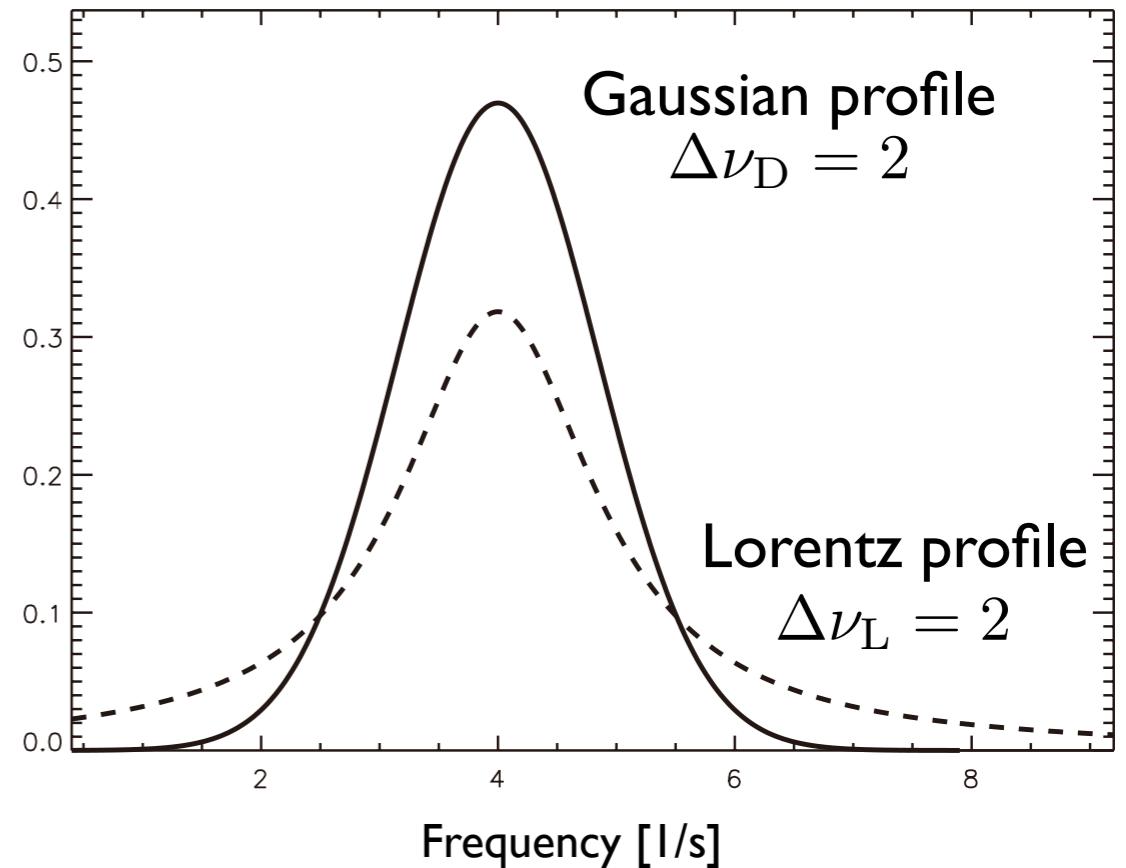
$$2\sigma^2 = \Delta\nu_D^2 / (4 \ln 2)$$

$$\Delta\nu_D = \frac{2\nu_0}{c} \left[ \ln 2 \left( \frac{2kT_k}{M} + V^2 \right) \right]^{1/2}$$

$$\Phi(\nu) = \frac{\Delta\nu_L/2\pi}{(\nu - \nu_0)^2 + (\Delta\nu_L/2)^2}$$

$$\Delta\nu_L = (2\pi\tau)^{-1} = [2\pi(\tau_N + \tau_R + \tau_C)]^{-1}$$

$\tau_N = 1/A_{ij}$     $\tau_C$  : the mean collision time



$$\tau_R = \frac{4m\nu_{mn}}{Ne^2f_a} \approx \frac{\nu_{mn}}{6 \times 10^7 N f_a} \text{ [s]}$$

# The Voigt Profile

## Gaussian Profile + Lorentz Profile

$$\Phi(\nu) = \int_{-\infty}^{+\infty} \frac{\Delta\nu_L \exp(-\Delta\nu^2/\Delta\nu_D^2) d(\Delta\nu)}{2\pi\sqrt{\pi}\Delta\nu_D \left[ (\nu - \nu_0 - \Delta\nu)^2 + (\Delta\nu_L/2)^2 \right]}$$

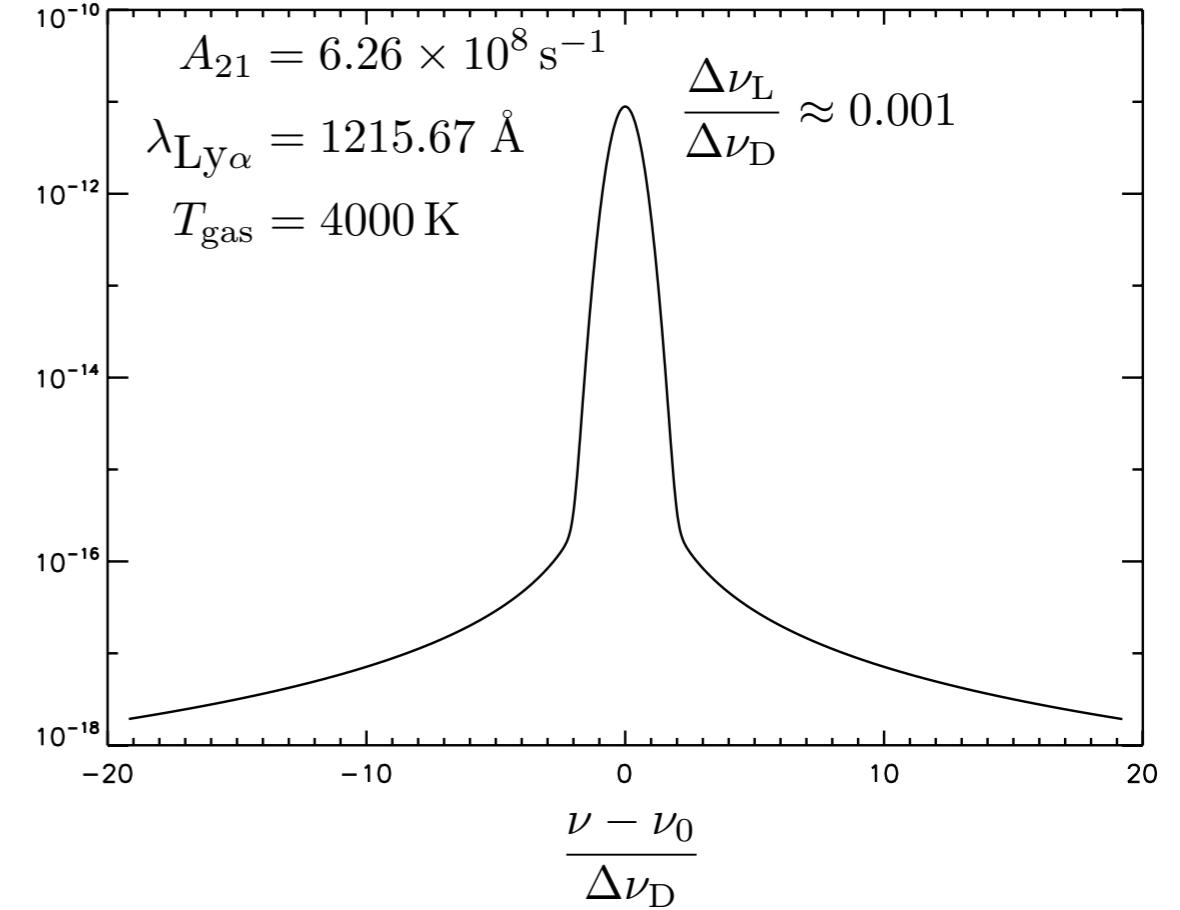
$$\Phi(\nu) = \frac{2\sqrt{\ln 2}}{\sqrt{\pi}\Delta\nu_D} H(a, b)$$

$$a = \sqrt{\ln 2}\Delta\nu_L / (2\Delta\nu_D)$$

$$b = 2\sqrt{\ln 2}(\nu - \nu_0) / \Delta\nu_D$$

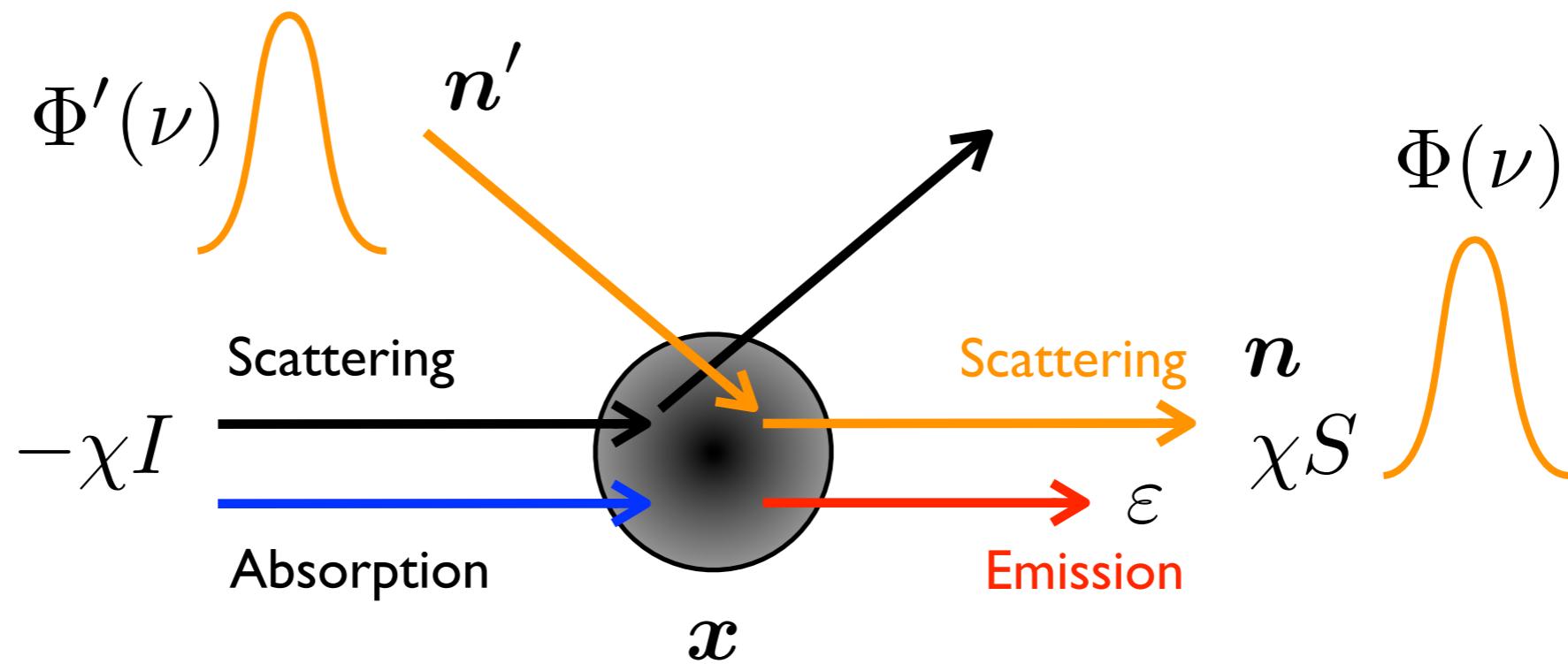
$$H(a, b) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-y^2) dy}{(b - y)^2 + a^2}$$

$$H(a, b) \approx \exp(-b^2) + \frac{a}{\sqrt{\pi}b^2}$$



# Scattering Processes

## Line Profile



$$\frac{1}{c} \frac{\partial I}{\partial t} + \mathbf{n} \cdot \nabla I = \left[ \chi_{\text{abs}} B + \chi_{\text{sca}} \int I(n') \phi(n, n') d\Omega \right] - (\chi_{\text{abs}} + \chi_{\text{sca}}) I$$

# Timescales

## Collision time .vs. Transition time

- Collision time for neutral Hydrogen atoms

$$t_{\text{coll}} \approx 10^{12} T^{-1/2} n_{\text{H}}^{-1} = 1.58 \times 10^{-3} \left[ \frac{T}{4000 \text{ [K]}} \right]^{-1/2} \left[ \frac{n_{\text{H}}}{10^{13} \text{ [1/cc]}} \right]^{-1} \text{ [sec]}$$

- Transition time of Hydrogen atom ( $2\text{p} \rightarrow 1\text{s}$ )

$$t_{2\text{p} \rightarrow 1\text{s}} = \frac{1}{A_{21}} = \frac{1}{6.26 \times 10^8 \text{ [s}^{-1}\text{]}} = 1.597 \times 10^{-9} \text{ [sec]}$$

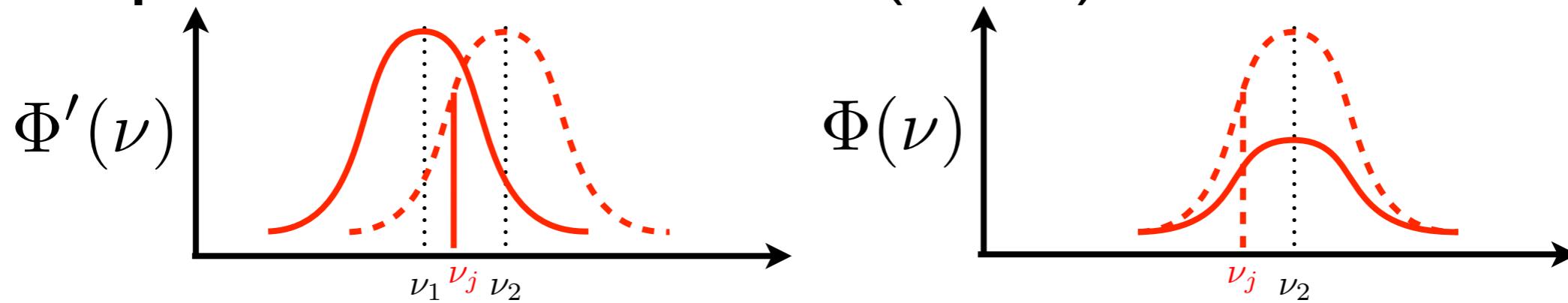
$$t_{\text{coll}} \gg t_{2\text{p} \rightarrow 1\text{s}}$$

# Frequency Redistribution

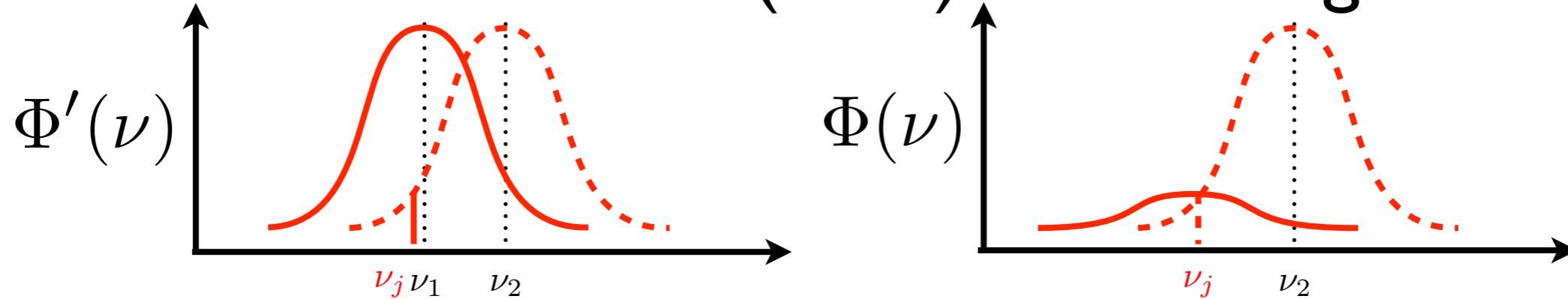
## Complete Re-Distribution (CRD) and Partial Re-Distribution (PRD)

- Atom 1  $\nu_1 \rightarrow$  Atom 2  $\nu_2$

- Complete Re-Distribution (CRD) in the core



- Partial Re-Distribution (PRD) in the wings



# Emergent Intensity Profile

**Complete Re-Distribution (CRD) and Partial Re-Distribution (PRD)**

- Complete Re-Distribution (CRD)

$$I(\nu_2) = \Phi_{\text{CRD}}(\nu_2) \sum_j I(\nu_j)$$

- Partial Re-Distribution (PRD)

$$I(\nu_2) = \sum_j \Phi_{\text{PRD}}(\nu_j) I(\nu_j)$$

# To be continued,,,

- ▶ Methods for Solving Radiation Transfer Equations
  - ✓ Mesh-based Radiation Transfer Solver
  - ✓ Monte-Carlo Radiation Transfer Solver
- ▶ Test Calculation of the Solar Model Atmosphere
  - ✓ Static Plane-Parallel Model Atmosphere
  - ✓ RMHD Model Atmosphere