Radiation transport and polarized line profile formation

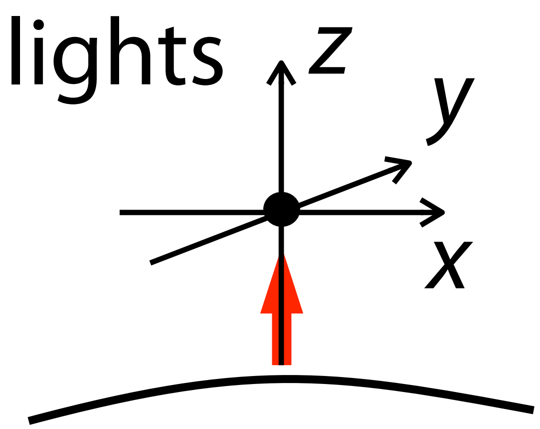
M. Goto, NIFS

intuitive understanding

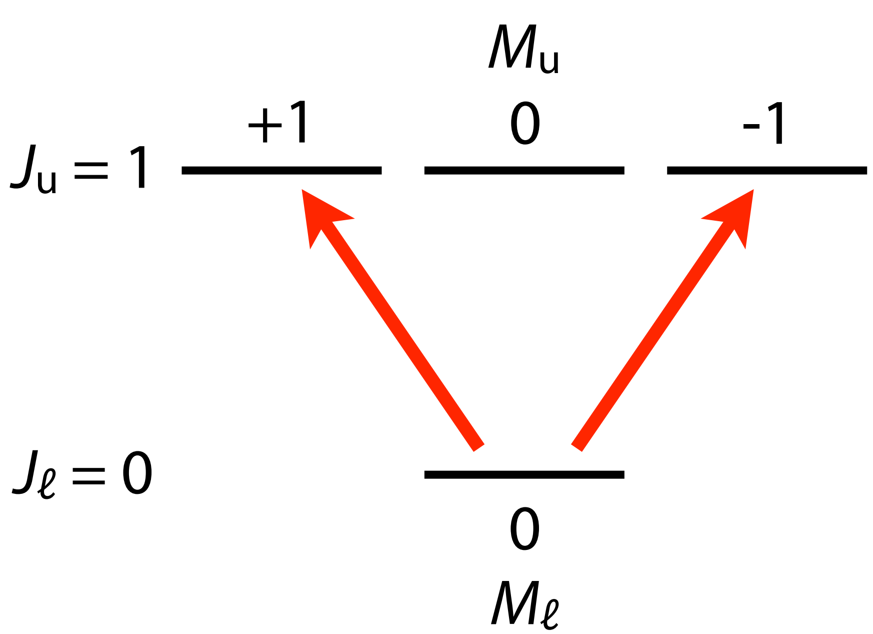
- two level atom ($J_\ell = 0$ and $J_u = 1$) is taken as example
- density matrix is set up for atoms under anisotropic irradiation (incoherent)
- rotation of coordinate gives rise to coherence between magnetic sublevels
- equation of motion due to magnetic field perturbation is solved for density matrix
- Stokes parameters are derived from density matrix

anisotropic photo-excitation

- unpolarized σ-light can be understood to consist of incoherent two circularly polarized lights



- excitation gives rise to anisotropic excited level

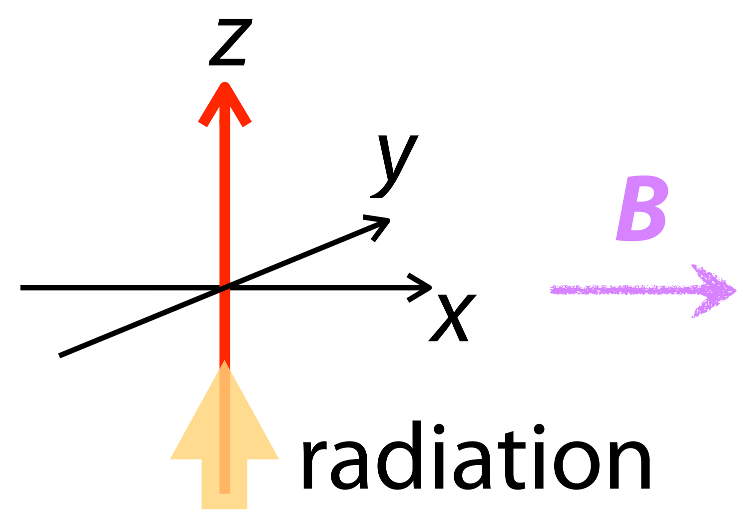


$$\Rightarrow \quad \rho = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} M_u = +1 & 0 & -1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} +1 \\ 0 \\ -1 \end{array}$$

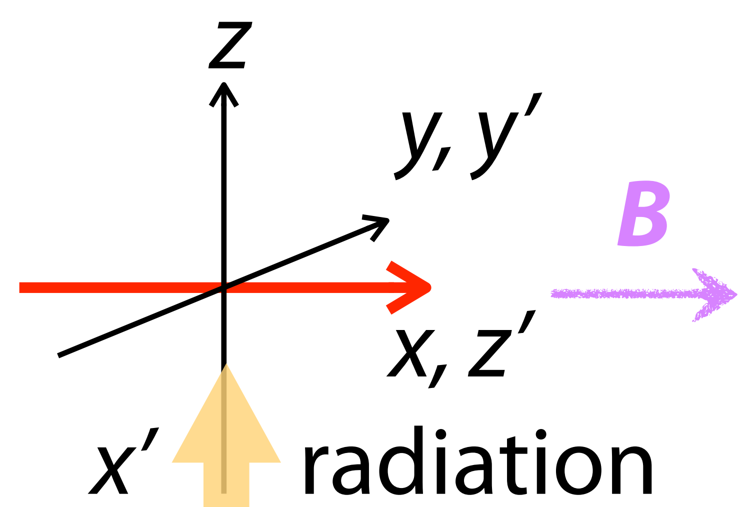
- there is no coherence (non-diagonal component) at this moment

- coordinates are rotated so that quantization axis points to ***B*** direction

$$\rho_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\rho_x = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



- coherence appears between $M = +1$ and $M = -1$ states

role of magnetic field

- equation of motion for density matrix

$$i\hbar \frac{\partial}{\partial t} \rho_x = [H_\mathrm{F}, \rho_x]$$

- Hamiltonian H_F consists of perturbation due to magnetic field

$$\begin{aligned}\langle M | H_\mathrm{F} | N \rangle &= -\mu_\mathrm{B} g_J B \langle M | J_x | N \rangle \\ &= -\mu_\mathrm{B} g_J B M \delta_{MN} \\ &= -\hbar \omega_0 M \delta_{MN}\end{aligned}$$

- μ_B and g_J are Bohr magneton and Landé g-factor, respectively, and ω_0 corresponds to Larmor angular frequency

- H_F is explicitly written as

$$H_\mathrm{F} = -\hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

equation of motion

$$i\hbar \frac{\partial}{\partial t}\rho_x = [H_\mathrm{F}, \rho_x]$$

- right hand side of equation of motion is calculated as

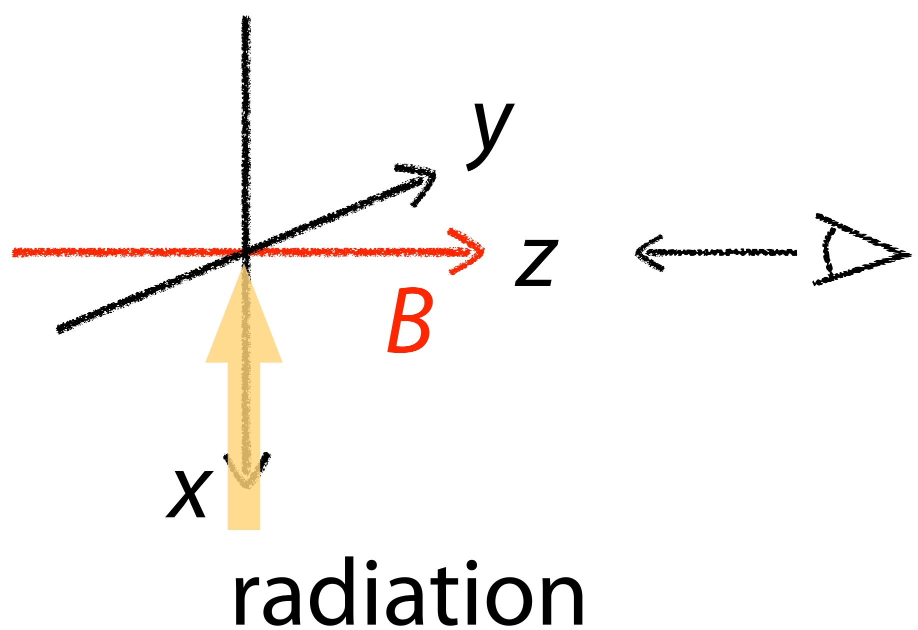
$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \rho_{1\,1} & \rho_{1\,0} & \rho_{1\,-1} \\ \rho_{0\,1} & \rho_{0\,0} & \rho_{0\,-1} \\ \rho_{-1\,1} & \rho_{-1\,0} & \rho_{-1\,-1} \end{pmatrix} = -\hbar\omega_0 \begin{pmatrix} 0 & \rho_{1\,0} & 2\rho_{1\,-1} \\ -\rho_{0\,1} & 0 & \rho_{0\,-1} \\ -2\rho_{-1\,1} & -\rho_{-1\,0} & 0 \end{pmatrix}$$

with $\rho_x(0) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

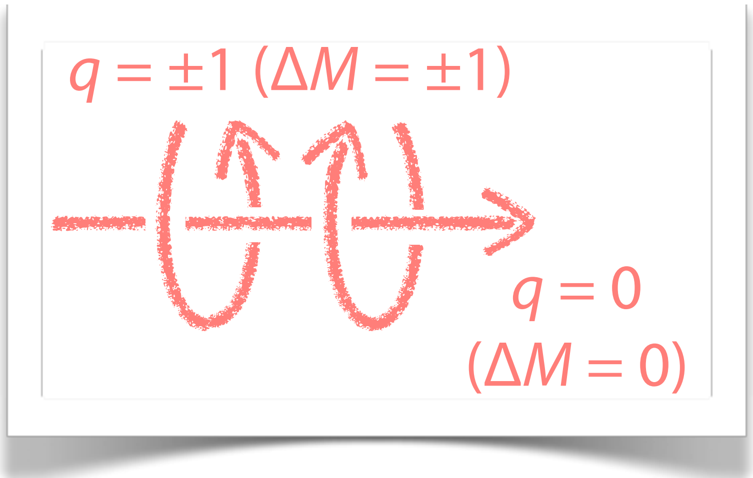
- $\rho_x(t)$ is readily obtained with initial condition

$$\rho_x(t) = \frac{1}{4}\begin{pmatrix} 1 & 0 & e^{2i\omega t} \\ 0 & 2 & 0 \\ e^{-2i\omega t} & 0 & 1 \end{pmatrix}$$

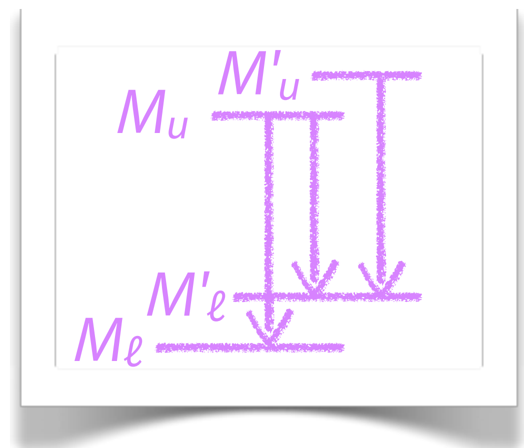
- line intensity is derived from density matrix obtained



line intensity



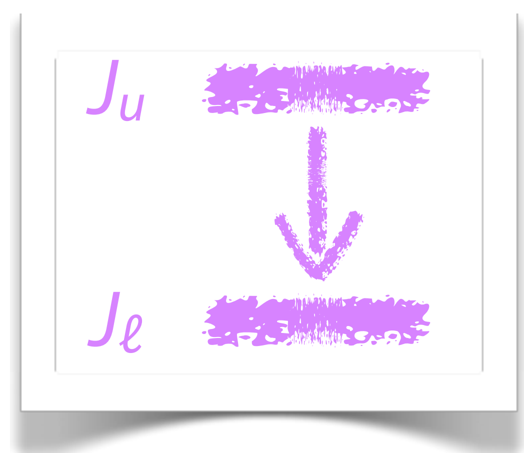
spherical components



$$I^q_{M_u M_\ell} = C_{\mathrm{D}} \left| \langle u J_u M_u | d_q | \ell J_\ell M_\ell \rangle \right|^2$$

$$\Downarrow$$

$$I^q_{u\ell} = C_{\mathrm{D}} \sum_{M_u, M_\ell} w_{M_u} \left| \langle u J_u M_u | d_q | \ell J_\ell M_\ell \rangle \right|^2$$



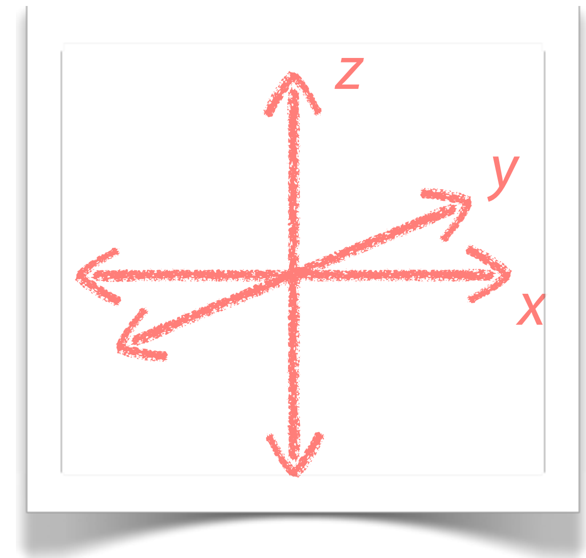
- measurable intensity is summation over all combinations of magnetic sublevels

linear polarization components

$d_q \rightarrow d_x$ and d_y

$$d_x = \frac{1}{\sqrt{2}}(d_{-1} - d_1)$$

$$d_y = \frac{\mathrm{i}}{\sqrt{2}}(d_{-1} + d_1)$$



$$I^x_{u\ell} = \frac{C_\mathrm{D}}{2} \sum_{M'_u, M''_u, M_\ell} \langle u J_u M'_u | \rho_u | u J_u M''_u \rangle \times \langle u J_u M''_u | d_{-1} - d_1 | \ell J_\ell M_\ell \rangle \langle \ell J_\ell M_\ell | d^\dagger_{-1} - d^\dagger_1 | u J_u M'_u \rangle$$

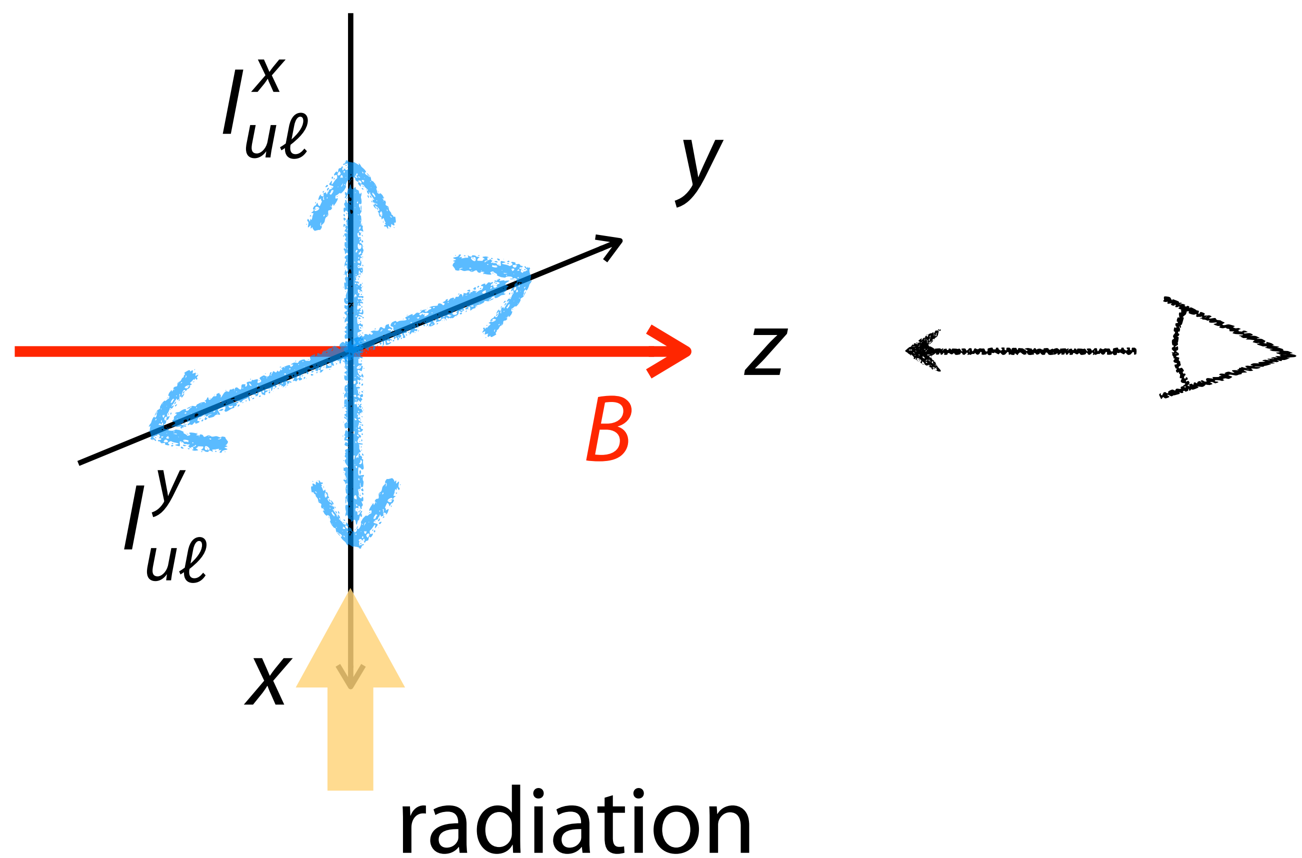
$$I^y_{u\ell} = \frac{C_\mathrm{D}}{2} \sum_{M'_u, M''_u, M_\ell} \langle u J_u M'_u | \rho_u | u J_u M''_u \rangle \times \langle u J_u M''_u | d_{-1} + d_1 | \ell J_\ell M_\ell \rangle \langle \ell J_\ell M_\ell | d^\dagger_{-1} + d^\dagger_1 | u J_u M'_u \rangle$$

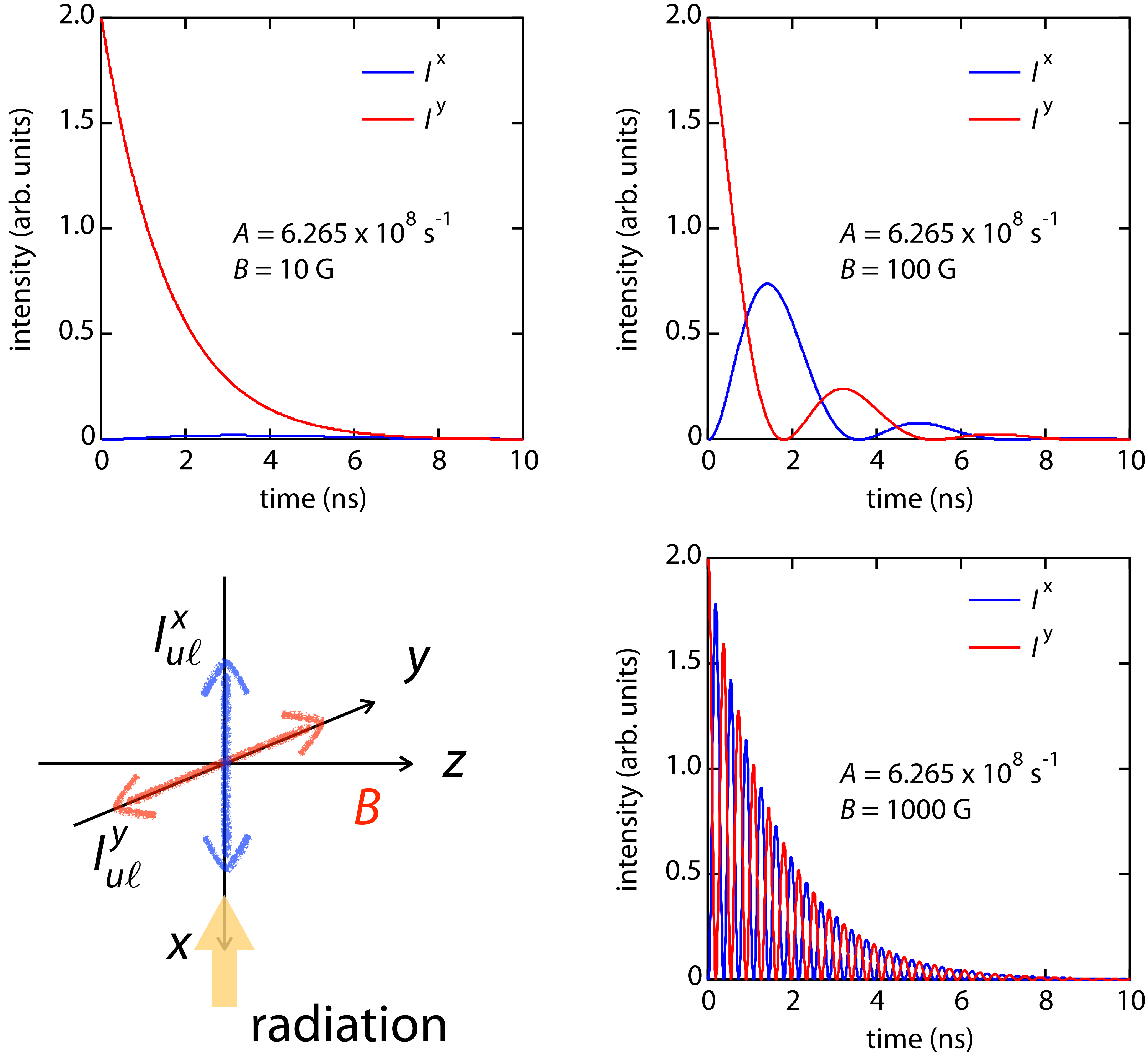
$$I^x_{u\ell} = \frac{C_D}{12}(1 - \cos 2\omega t)|\langle u1\|\mathbf{d}\|\ell 0\rangle|^2 \times \exp(-A_{\alpha\beta}t)$$

$$I^y_{u\ell} = \frac{C_D}{12}(1 + \cos 2\omega t)|\langle u1\|\mathbf{d}\|\ell 0\rangle|^2 \times \exp(-A_{\alpha\beta}t)$$

$$\omega = \frac{e}{2m} g_J B$$

spontaneous decay




intensity (arb. units)
time (ns)
I^x
I^y
A = 6.265 x 10^8 s^{-1}
B = 10 G
B = 100 G
B = 1000 G
$I^x_{u\ell}$
$I^y_{u\ell}$
y
z
x
B
radiation

statistical equilibrium equations

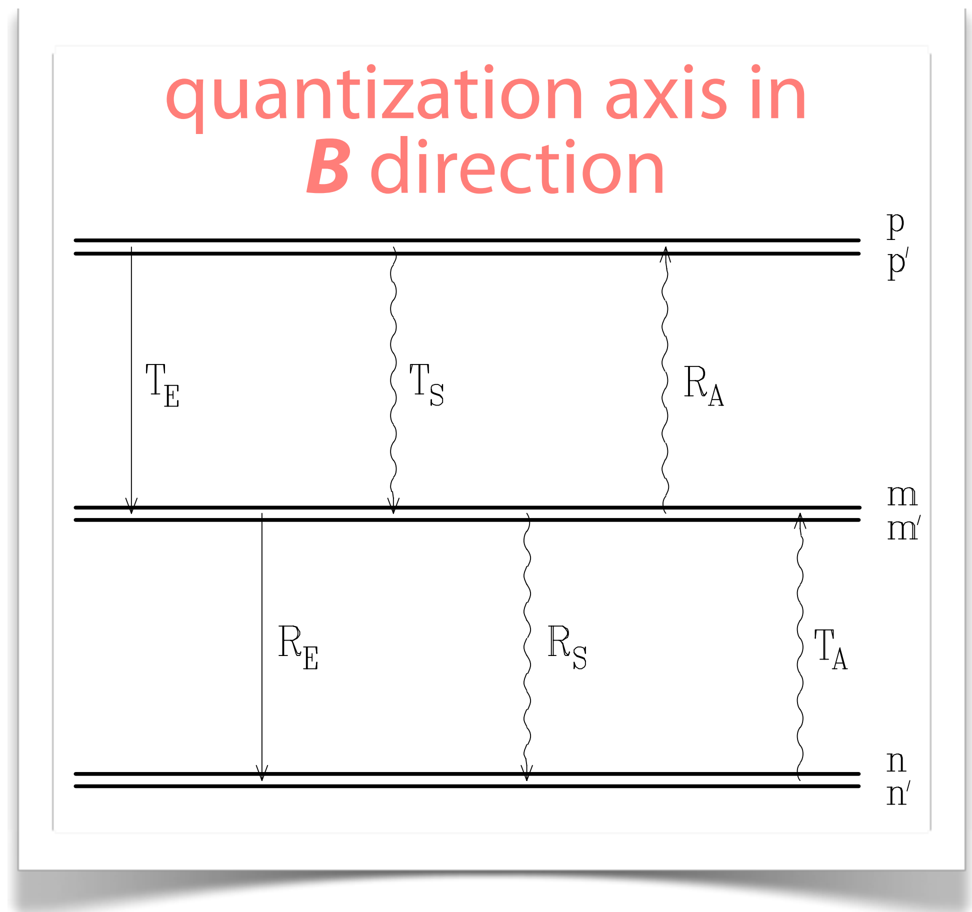
- density matrix and Stokes parameters are derived following "*Polarization in Spectral Lines*" by E. Landi Degl'Innocenti and M. Landolfi
- correspondence to the intuitive method of the results is considered

equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho = \frac{2\pi}{\mathrm{i}h}\,[H,\rho]$$

- Hamiltonian can involve atomic processes in addition to magnetic field

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\rho_{\alpha J}(M,M') = {} & -2\pi\mathrm{i}\,\nu_{\mathrm{L}}\,g_{\alpha J}\,(M-M')\,\rho_{\alpha J}(M,M') \\
& + \sum_{\alpha_\ell J_\ell}\,\sum_{M_\ell M'_\ell}\,\rho_{\alpha_\ell J_\ell}(M_\ell,M'_\ell)\,T_{\mathrm{A}}(\alpha J M M',\alpha_\ell J_\ell M_\ell M'_\ell) \\
& + \sum_{\alpha_u J_u}\,\sum_{M_u M'_u}\,\rho_{\alpha_u J_u}(M_u,M'_u)\,\Big[T_{\mathrm{E}}(\alpha J M M',\alpha_u J_u M_u M'_u) \\
& \qquad\qquad + T_{\mathrm{S}}(\alpha J M M',\alpha_u J_u M_u M'_u)\Big] \\
& - \sum_{M''}\Big\{\rho_{\alpha J}(M,M'')\Big[R_{\mathrm{A}}(\alpha J M' M'') + R_{\mathrm{E}}(\alpha J M'' M') \\
& \qquad\qquad + R_{\mathrm{S}}(\alpha J M'' M')\Big] \\
& \qquad + \rho_{\alpha J}(M'',M')\Big[R_{\mathrm{A}}(\alpha J M'' M) + R_{\mathrm{E}}(\alpha J M M'') \\
& \qquad\qquad + R_{\mathrm{S}}(\alpha J M M'')\Big]\Big\}
\end{aligned}$$



← standard representation

spherical tensors

- spherical representation of density matrix is obtained from standard matrix as

$$\rho^K_Q(\alpha J, \alpha J) = \rho^K_Q(\alpha J)$$

$$= \sum_{MM'} (-1)^{J-M} \sqrt{2K+1} \begin{pmatrix} J & J & K \\ M & -M' & -Q \end{pmatrix} \rho_{\alpha J}(M, M')$$

where $K = 0, 1, \ldots, 2J$ and $Q = -K, \ldots, K$

- the transformation is understood as change of matrix *basis*, e.g., for $J = 1/2$

$$\rho_{\alpha J}(M, M') : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho^K_Q(\alpha J) : \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

advantages

- standard representation requires two rotation matrices in rotation of coordinates,

$$\Big[\rho_{\alpha J}(M, M')\Big]_{\text{new}} = \sum_{NN'} \underline{\mathcal{D}^J_{NM}(R)^* \, \mathcal{D}^J_{N'M'}(R)} \, \Big[\rho_{\alpha J}(N, N')\Big]_{\text{old}}$$

  while spherical representation needs just one rotation matrix

$$\Big[\rho^K_Q(\alpha J, \alpha' J')\Big]_{\text{new}} = \sum_{Q'} \Big[\rho^K_{Q'}(\alpha J, \alpha' J')\Big]_{\text{old}} \, \underline{\mathcal{D}^K_{Q'Q}(R)^*}$$

- many components vanish when there exists some symmetry

spherical representation

- multiplying both sides in equation of motion by

$$(-1)^{J-M}\sqrt{2K+1}\begin{pmatrix} J & J & K \\ M & -M' & -Q \end{pmatrix}$$

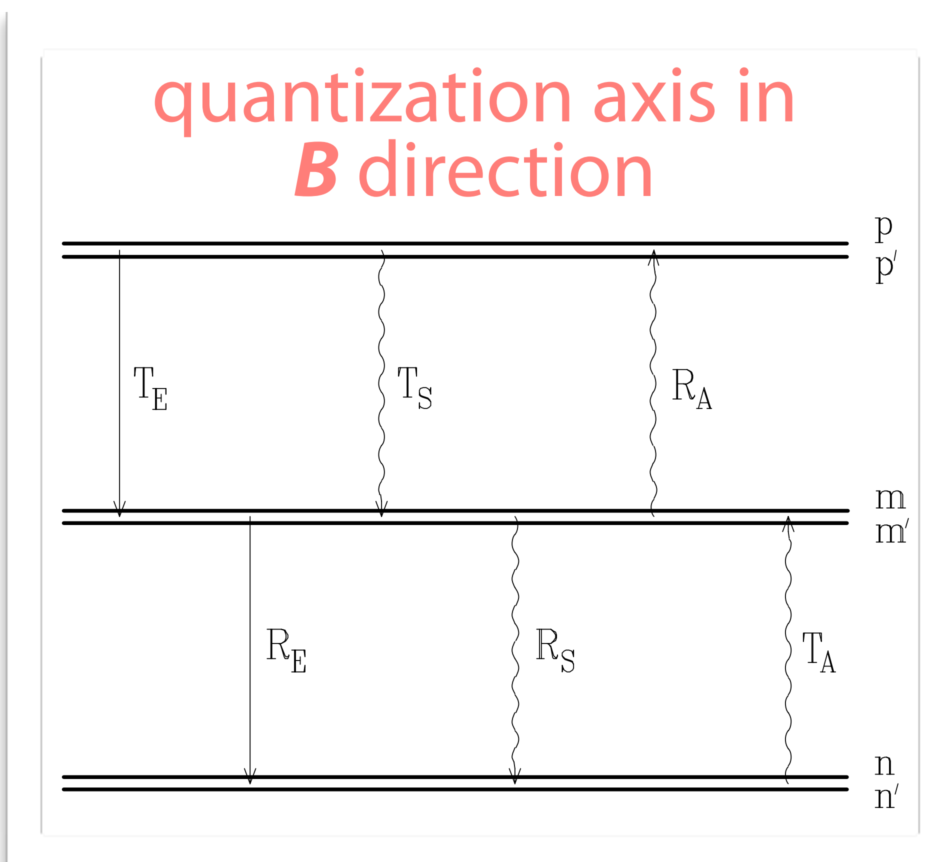
and carrying out summation over M and M' give

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho^K_Q(\alpha J) = -2\pi\mathrm{i}\,\nu_{\mathrm{L}}\;g_{\alpha J}\;Q\;\rho^K_Q(\alpha J)$$

$$+\sum_{\alpha_\ell J_\ell}\sum_{K_\ell Q_\ell}\rho^{K_\ell}_{Q_\ell}(\alpha_\ell J_\ell)\;\mathbb{T}_{\mathrm{A}}(\alpha JKQ,\alpha_\ell J_\ell K_\ell Q_\ell)+$$

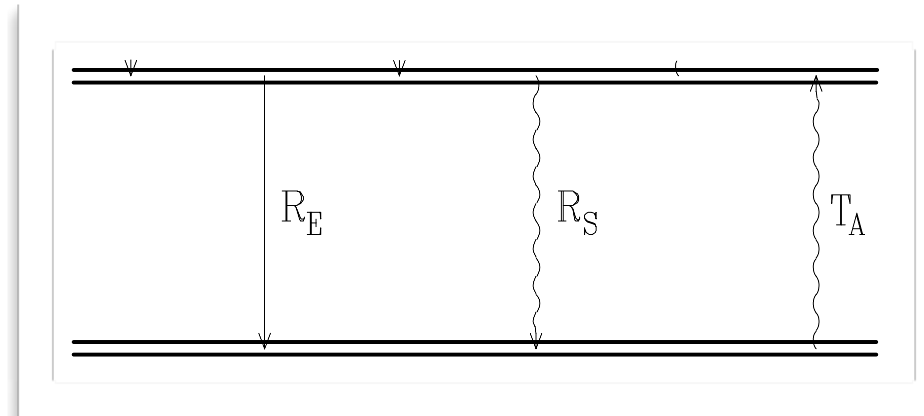
$$+\sum_{\alpha_u J_u}\sum_{K_u Q_u}\rho^{K_u}_{Q_u}(\alpha_u J_u)\Big[\mathbb{T}_{\mathrm{E}}(\alpha JKQ,\alpha_u J_u K_u Q_u) + \mathbb{T}_{\mathrm{S}}(\alpha JKQ,\alpha_u J_u K_u Q_u)\Big]$$

$$-\sum_{K'Q'}\rho^{K'}_{Q'}(\alpha J)\Big[\mathbb{R}_{\mathrm{A}}(\alpha JKQK'Q')+\mathbb{R}_{\mathrm{E}}(\alpha JKQK'Q') + \mathbb{R}_{\mathrm{S}}(\alpha JKQK'Q')\Big]$$



two-level atom

- upper level



$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho^K_Q(\alpha_u J_u) = -2\pi\mathrm{i}\,\nu_{\mathrm{L}}\;g_{\alpha_u J_u}\;Q\;\rho^K_Q(\alpha_u J_u)$$

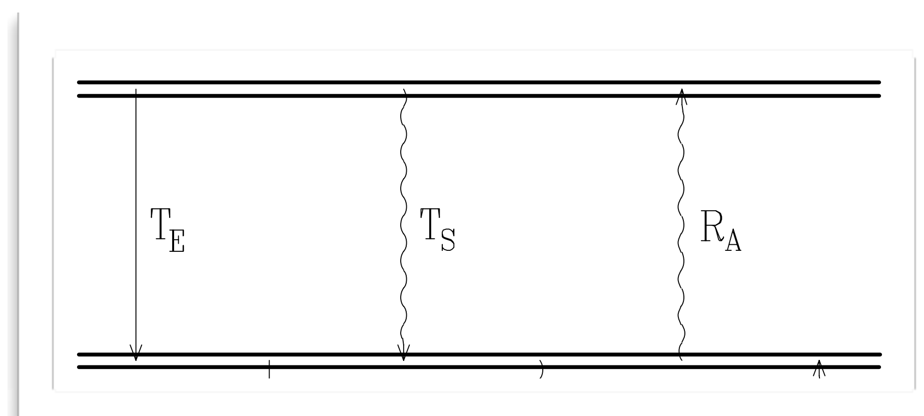
$$+\sum_{K'Q'}\mathbb{T}_{\mathrm{A}}(\alpha_u J_u K Q, \alpha_\ell J_\ell K' Q')\;\rho^{K'}_{Q'}(\alpha_\ell J_\ell)$$

$$-\sum_{K'Q'}\Big[\underline{\mathbb{R}_{\mathrm{E}}(\alpha_u J_u K Q K' Q')} + \underline{\mathbb{R}_{\mathrm{S}}(\alpha_u J_u K Q K' Q')}\Big]\;\rho^{K'}_{Q'}(\alpha_u J_u)$$

$$\delta_{KK'}\,\delta_{QQ'}\sum_{\alpha_\ell J_\ell} A(\alpha J \rightarrow \alpha_\ell J_\ell)$$

ignored

- lower level



$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho^K_Q(\alpha_\ell J_\ell) = -2\pi\mathrm{i}\,\nu_{\mathrm{L}}\;g_{\alpha_\ell J_\ell}\;Q\;\rho^K_Q(\alpha_\ell J_\ell)$$

$$+\sum_{K'Q'}\Big[\mathbb{T}_{\mathrm{E}}(\alpha_\ell J_\ell K Q, \alpha_u J_u K' Q') + \mathbb{T}_{\mathrm{S}}(\alpha_\ell J_\ell K Q, \alpha_u J_u K' Q')\Big]\;\rho^{K'}_{Q'}(\alpha_u J_u)$$

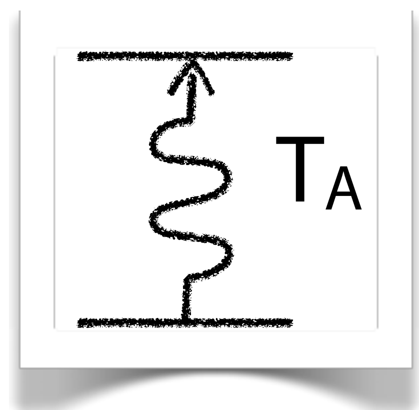
$$-\sum_{K'Q'}\mathbb{R}_{\mathrm{A}}(\alpha_\ell J_\ell K Q K' Q')\;\rho^{K'}_{Q'}(\alpha_\ell J_\ell)$$

- when stationary and lower level is unpolarized

only this term remains

$$\rho^K_Q(\alpha_u J_u) = \frac{\mathbb{T}_\mathrm{A}(\alpha_u J_u K Q, \alpha_\ell J_\ell \, 0 \, 0)}{2\pi \mathrm{i} \nu_\mathrm{L} g_{\alpha_u J_u} Q + A(\alpha_u J_u \to \alpha_\ell J_\ell)} \times \rho^0_0(\alpha_\ell J_\ell)$$

$$\mathbb{T}_\mathrm{A}(\alpha J K Q, \alpha_\ell J_\ell \cancelto{0}{K_\ell} \cancelto{0}{Q_\ell}) = (2J_\ell + 1)\, B(\alpha_\ell J_\ell \to \alpha J)$$
$$\times \sum_{K_\mathrm{r} Q_\mathrm{r}} \sqrt{3(2K+1)(2\cancelto{0}{K_\ell}+1)(2K_\mathrm{r}+1)}$$
$$\times (-1)^{\cancelto{0}{K_\ell} + \cancelto{0}{Q_\ell}} \begin{Bmatrix} J & J_\ell & 1 \\ J & J_\ell & 1 \\ K & \cancelto{0}{K_\ell} & K_\mathrm{r} \end{Bmatrix} \underbrace{\begin{pmatrix} K & \cancelto{0}{K_\ell} & K_\mathrm{r} \\ -Q & \cancelto{0}{Q_\ell} & -Q_\mathrm{r} \end{pmatrix}}_{K_\mathrm{r} = K,\ Q_\mathrm{r} = -Q} \underbrace{J^{K_\mathrm{r}}_{Q_\mathrm{r}}(\nu_{\alpha J, \alpha_\ell J_\ell})}_{\text{radiation field tensor}}$$



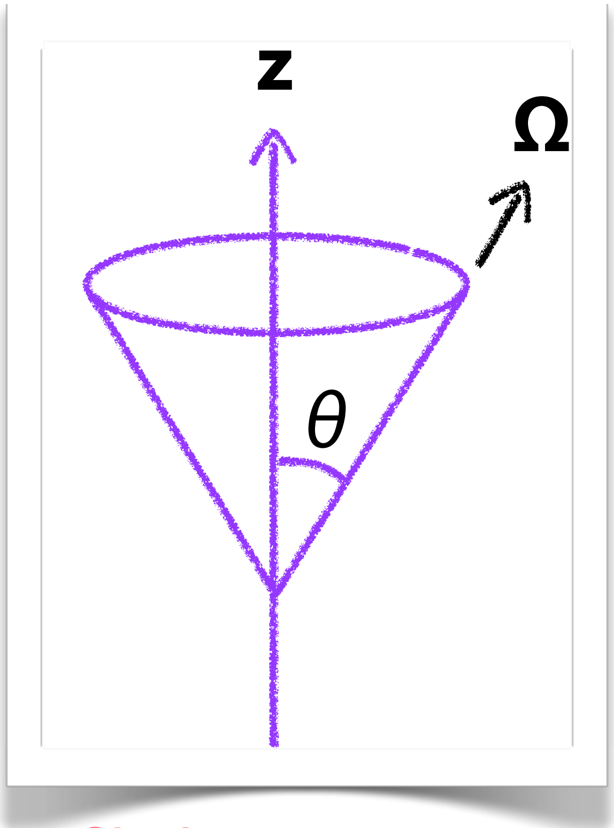
$$
\begin{aligned}
&\mathbb{T}_{\mathrm{A}}(\alpha_u J_u K Q, \alpha_\ell J_\ell\, 0\, 0) \\
&\quad = (2J_\ell + 1) B(\alpha_\ell J_\ell \to \alpha_u J_u) \times \sqrt{3(2K+1)^2} \\
&\qquad \times \begin{Bmatrix} J_u & J_\ell & 1 \\ J_u & J_\ell & 1 \\ K & 0 & K \end{Bmatrix} \begin{pmatrix} K & 0 & K \\ -Q & 0 & Q \end{pmatrix} J^K_{-Q}(\nu_{\alpha_u J_u, \alpha_\ell J_\ell})
\end{aligned}
$$

$$
\begin{aligned}
&\quad = \sqrt{3(2J_\ell + 1)} B(\alpha_\ell J_\ell \to \alpha_u J_u) \\
&\qquad \times (-1)^{1+J_u+J_\ell+Q} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix} J^K_{-Q}(\nu_{\alpha_u J_u, \alpha_\ell J_\ell})
\end{aligned}
$$

Stokes parameters

$$
J^K_Q(\nu) = \oint \frac{\mathrm{d}\Omega}{4\pi}\, \mathcal{I}^K_Q(\nu, \vec{\Omega}) = \oint \frac{\mathrm{d}\Omega}{4\pi} \sum_{i=0}^{3} \mathcal{T}^K_Q(i, \vec{\Omega})\, S_i(\nu, \vec{\Omega})
$$

geometrical factors



for unpolarized radiation having *z*-axis symmetry

$$
J^0_0(\nu) = \oint \frac{\mathrm{d}\Omega}{4\pi}\, I(\nu, \theta)
$$

$$
J^2_0(\nu) = \frac{1}{2\sqrt{2}} \oint \frac{\mathrm{d}\Omega}{4\pi}\, (3\cos^2\theta - 1)\, I(\nu, \theta)
$$
(= 0 for isotropic filed)

$$\rho^K_Q(\alpha_u J_u) = \sqrt{\frac{2J_\ell + 1}{2J_u + 1}}\ \frac{B(\alpha_\ell J_\ell \to \alpha_u J_u)}{A(\alpha_u J_u \to \alpha_\ell J_\ell) + 2\pi \mathrm{i}\, \nu_\mathrm{L}\ g_{\alpha_u J_u}\ Q} \times w^{(K)}_{J_u J_\ell}\, (-1)^Q\ J^K_{-Q}(\nu_0)\ \ \rho^0_0(\alpha_\ell J_\ell)$$

$$w^{(K)}_{J_u J_\ell} = (-1)^{1+J_\ell+J_u} \sqrt{3(2J_u+1)} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix}$$

$$\boxed{\rho^K_Q(\alpha_u J_u) = \frac{1}{1 + \mathrm{i}\, Q H_u} \left[\rho^K_Q(\alpha_u J_u)\right]_{B=0}}$$

essence of Hanle effect

$$H_u = \frac{2\pi \nu_\mathrm{L}\ g_{\alpha_u J_u}}{A(\alpha_u J_u \to \alpha_\ell J_\ell)}$$

$$\rho^K_Q(\alpha_u J_u) = \frac{1}{1 + \mathrm{i}\, Q H_u} \left[\rho^K_Q(\alpha_u J_u)\right]_{B=0}$$

- if QH_u is expressed to be $\tan(\alpha)$, $\rho^K_Q(\alpha_u J_u)$ is rewritten as

$$\rho^K_Q(\alpha_u J_u) = \mathrm{e}^{-\mathrm{i}\alpha} \cos\alpha \left[\rho^K_Q(\alpha_u J_u)\right]_{B=0}$$

- effect of magnetic field is to reduce by factor of

$$\cos\alpha = \sqrt{\frac{1}{1 + Q^2 H_u^2}}$$

and to dephase by

$$\tan^{-1} Q H_u$$

e.g.

$$\rho_x(t) = \frac{1}{4} \begin{pmatrix} 1 & 0 & \mathrm{e}^{2\mathrm{i}\omega t} \\ 0 & 2 & 0 \\ \mathrm{e}^{-2\mathrm{i}\omega t} & 0 & 1 \end{pmatrix}$$

application: stellar atmospheres

- two-level model (J_ℓ, J_u)
- negligible stimulated emission due to weak radiation field
- isotropic lower level, i.e., $\rho^K_Q(\alpha_\ell J_\ell) = \rho^0_0(\alpha_\ell J_\ell)\delta_{K0}\delta_{Q0}$
- complete redistribution

$$J^K_Q(\nu_0) \to \bar{J}^K_Q(\nu_0) = \int_{-\infty}^{\infty} \mathrm{d}\nu \; p(\nu_0 - \nu) \; J^K_Q(\nu)$$

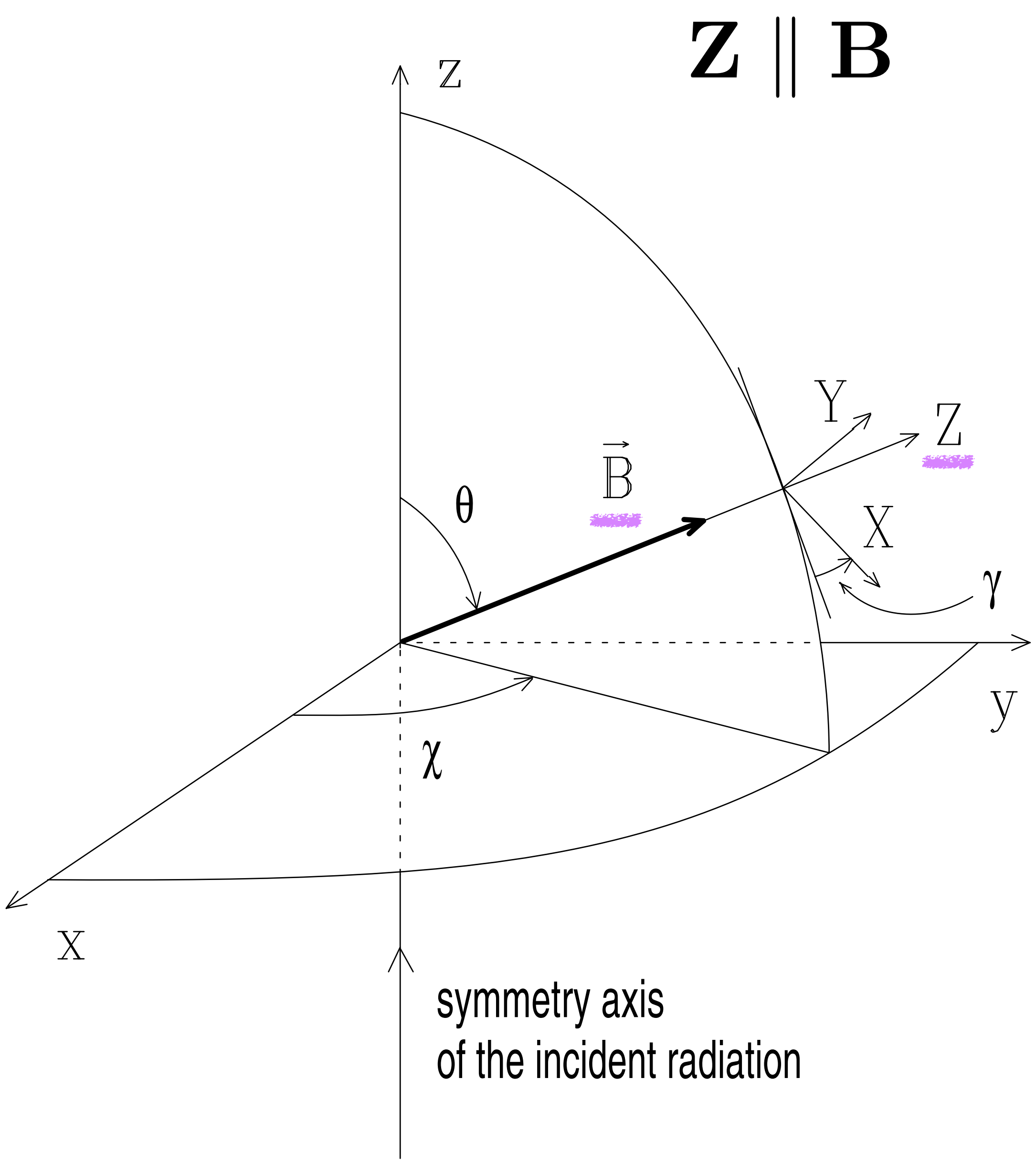
- plane-parallel atmosphere

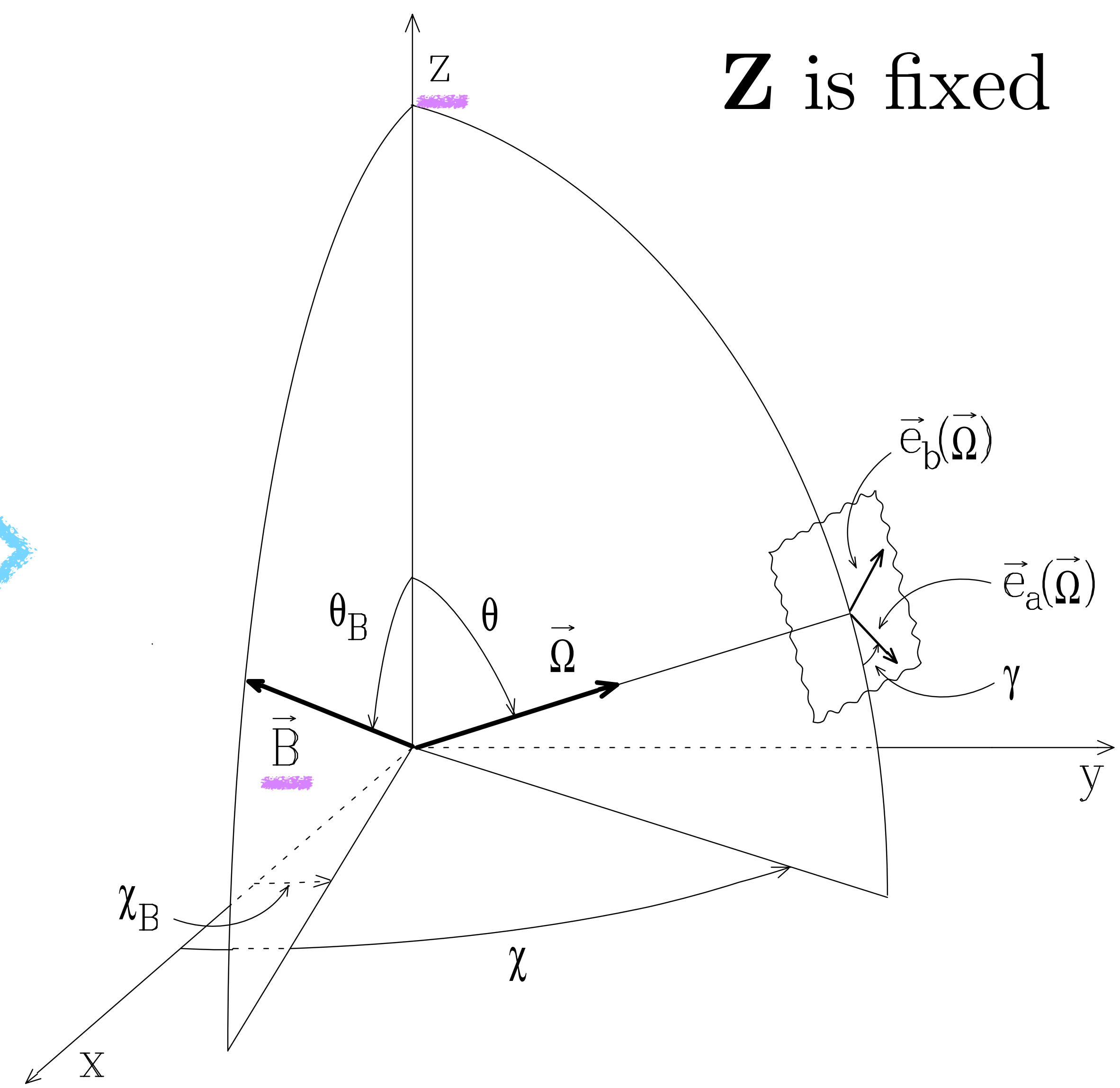
$$k^{\mathrm{A}}_{\mathrm{L}} = k^{\mathrm{A}}_{\mathrm{L}} \int p(\nu_0 - \nu)\mathrm{d}\nu = (k^{\mathrm{A}}_{\mathrm{L}}\bar{p})\Delta\nu_{\mathrm{D}}$$

$$f(\vec{x}) \to f(z) \qquad \mathrm{d}t_{\mathrm{L}} = -\frac{k^{\mathrm{A}}_{\mathrm{L}}}{\Delta\nu_{\mathrm{D}}}\mathrm{d}z$$

t_{L}: line optical depth

fixed quantization axis





SEE with fixed Z-axis

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho^K_Q(\alpha_u J_u)\right]_{\vec{x}} = & -2\pi\mathrm{i}\,\nu_\mathrm{L}\,g_{\alpha_u J_u}\underbrace{\sum_{Q'}\mathcal{K}^K_{QQ'}\left[\rho^K_{Q'}(\alpha_u J_u)\right]_{\vec{x}}}_{\textbf{rotation}} \\
& + \sum_{K'Q'}\underbrace{\mathbb{T}_\mathrm{A}(\alpha_u J_u KQ,\alpha_\ell J_\ell K'Q')}_{\textbf{absorption}}\left[\rho^{K'}_{Q'}(\alpha_\ell J_\ell)\right]_{\vec{x}} + \\
& - \sum_{K'Q'}\Big[\underbrace{\mathbb{R}_\mathrm{E}(\alpha_u J_u KQK'Q')}_{\textbf{emission}} + \underbrace{\cancel{\mathbb{R}_\mathrm{S}(\alpha_u J_u KQK'Q')}}_{\textbf{stimulated emission}}\Big]\left[\rho^{K'}_{Q'}(\alpha_u J_u)\right]_{\vec{x}} \\
& \left.\begin{aligned}
& + \sqrt{\frac{2J_\ell+1}{2J_u+1}}\,\underbrace{C^{(K)}_\mathrm{I}(\alpha_u J_u,\alpha_\ell J_\ell)}_{\textbf{excitation}}\left[\rho^K_Q(\alpha_\ell J_\ell)\right]_{\vec{x}} \\
& - \Big[\underbrace{C^{(0)}_\mathrm{S}(\alpha_\ell J_\ell,\alpha_u J_u)}_{\textbf{de-excitation}} + \underbrace{D^{(K)}(\alpha_u J_u)}_{\textbf{de-polarization}}\Big]\left[\rho^K_Q(\alpha_u J_u)\right]_{\vec{x}}\,,
\end{aligned}\right\}\ \text{collisions}
\end{aligned}
$$

emission and absorption

$$\mathbb{R}_{\mathrm{E}}(\alpha J K Q K' Q') = \delta_{KK'}\,\delta_{QQ'} \sum_{\alpha_\ell J_\ell} A(\alpha J \to \alpha_\ell J_\ell)$$

Einstein's
A coefficient

$$\mathbb{T}_{\mathrm{A}}(\alpha_u J_u K Q, \alpha_\ell J_\ell\, 0\, 0) = \sqrt{3(2J_\ell+1)}\; B(\alpha_\ell J_\ell \to \alpha_u J_u) \times (-1)^{1+J_\ell+J_u+Q} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix} J^K_{-Q}(\nu_0)$$

radiation field
tensor

$$J^K_Q(\nu_0) \to \bar{J}^K_Q(\nu_0) = \int_{-\infty}^{\infty} \mathrm{d}\nu\;\, p(\nu_0-\nu)\; J^K_Q(\nu)$$

complete redistribution

expression of SEE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho^K_Q(\alpha_u J_u)\right]_{\vec{x}} = -2\pi\mathrm{i}\,\nu_\mathrm{L}\,g_{\alpha_u J_u}\sum_{Q'}\mathcal{K}^K_{QQ'}\left[\rho^K_{Q'}(\alpha_u J_u)\right]_{\vec{x}}$$
$$-\left[A(\alpha_u J_u\to\alpha_\ell J_\ell)+C^{(0)}_\mathrm{S}(\alpha_\ell J_\ell,\alpha_u J_u)+D^{(K)}(\alpha_u J_u)\right]\left[\rho^K_Q(\alpha_u J_u)\right]_{\vec{x}}$$
$$+\sqrt{\frac{2J_\ell+1}{2J_u+1}}\left[B(\alpha_\ell J_\ell\to\alpha_u J_u)\,w^{(K)}_{J_u J_\ell}\,(-1)^Q\bar{J}^K_{-Q}(\nu_0)\right.$$
$$\left.+\,\delta_{K0}\,\delta_{Q0}\,C^{(0)}_\mathrm{I}(\alpha_u J_u,\alpha_\ell J_\ell)\right]\left[\rho^0_0(\alpha_\ell J_\ell)\right]_{\vec{x}}\,,$$

with $w^{(K)}_{J_u J_\ell} = (-1)^{1+J_\ell+J_u}\sqrt{3(2J_u+1)}\begin{Bmatrix}1 & 1 & K\\ J_u & J_u & J_\ell\end{Bmatrix}$

$$C^{(0)}_\mathrm{I}(\alpha_u J_u,\alpha_\ell J_\ell) = \frac{2J_u+1}{2J_\ell+1}\,\mathrm{e}^{-\frac{h\nu_0}{k_\mathrm{B}T_\mathrm{c}}}\,C^{(0)}_\mathrm{S}(\alpha_\ell J_\ell,\alpha_u J_u)$$

Einstein-Milne relation

$$\left[1+\epsilon+\delta_u^{(K)}\right]\left[\rho_Q^K(\alpha_u J_u)\right]_{\vec{x}} + \mathrm{i}H_u \sum_{Q'} \mathcal{K}_{QQ'}^K \left[\rho_{Q'}^K(\alpha_u J_u)\right]_{\vec{x}} =$$

$$= \frac{c^2}{2h\nu_0^3}\sqrt{\frac{2J_u+1}{2J_\ell+1}}\left[w_{J_u J_\ell}^{(K)}\,(-1)^Q\,\bar{J}_{-Q}^K(\nu_0) + \delta_{K0}\,\delta_{Q0}\,\epsilon\,B_{\mathrm{P}}(T)\right]\left[\rho_0^0(\alpha_\ell J_\ell)\right]_{\vec{x}}\,,$$

where

$$\epsilon = \frac{C_{\mathrm{S}}^{(0)}(\alpha_\ell J_\ell, \alpha_u J_u)}{A(\alpha_u J_u \to \alpha_\ell J_\ell)}\,, \quad \delta_u^{(K)} = \frac{D^{(K)}(\alpha_u J_u)}{A(\alpha_u J_u \to \alpha_\ell J_\ell)}\,, \quad H_u = \frac{2\pi\nu_{\mathrm{L}}\, g_{\alpha_u J_u}}{A(\alpha_u J_u \to \alpha_\ell J_\ell)}\,, \quad B_{\mathrm{P}}(T) = \frac{2h\nu_0^3}{c^2}\ \mathrm{e}^{-\frac{h\nu_0}{k_{\mathrm{B}}T}}$$

$$\left[1+\epsilon+\delta_u^{(K)}\right] S_Q^K(\vec{x}\,) + \mathrm{i}H_u \sum_{Q'} \mathcal{K}_{QQ'}^K\, S_{Q'}^K(\vec{x}\,) =$$

$$= w_{J_u J_\ell}^{(K)}\,(-1)^Q\,\bar{J}_{-Q}^K(\nu_0) + \delta_{K0}\,\delta_{Q0}\,\epsilon\,B_{\mathrm{P}}(T)\,.$$

with $S_Q^K(\vec{x}\,) = \dfrac{2h\nu_0^3}{c^2}\sqrt{\dfrac{2J_\ell+1}{2J_u+1}}\ \dfrac{\left[\rho_Q^K(\alpha_u J_u)\right]_{\vec{x}}}{\left[\rho_0^0(\alpha_\ell J_\ell)\right]_{\vec{x}}}\,,$

kind of source function

radiative transfer equation

$$\frac{\mathrm{d}}{\mathrm{d}s} I_i(\nu, \vec{\Omega}) = -\sum_{j=0}^{3} K^{\mathrm{A}}_{ij}\; I_j(\nu, \vec{\Omega}) \;+\; \varepsilon_i \qquad (i = 0, \ldots, 3)$$

propagation matrix

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = - \begin{pmatrix} \eta^{\mathrm{A}}_I & \eta^{\mathrm{A}}_Q & \eta^{\mathrm{A}}_U & \eta^{\mathrm{A}}_V \\ \eta^{\mathrm{A}}_Q & \eta^{\mathrm{A}}_I & \rho^{\mathrm{A}}_V & -\rho^{\mathrm{A}}_U \\ \eta^{\mathrm{A}}_U & -\rho^{\mathrm{A}}_V & \eta^{\mathrm{A}}_I & \rho^{\mathrm{A}}_Q \\ \eta^{\mathrm{A}}_V & \rho^{\mathrm{A}}_U & -\rho^{\mathrm{A}}_Q & \eta^{\mathrm{A}}_I \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} + \begin{pmatrix} \varepsilon_I \\ \varepsilon_Q \\ \varepsilon_U \\ \varepsilon_V \end{pmatrix}$$

lower level
is isotropic

$$\begin{pmatrix} \eta^{\mathrm{A}}_I & 0 & 0 & 0 \\ 0 & \eta^{\mathrm{A}}_I & 0 & 0 \\ 0 & 0 & \eta^{\mathrm{A}}_I & 0 \\ 0 & 0 & 0 & \eta^{\mathrm{A}}_I \end{pmatrix}$$

$$\eta^{\mathrm{A}}_I = k^{\mathrm{A}}_{\mathrm{L}}\, p(\nu_0 - \nu)$$

$$\varepsilon_i = k^{\mathrm{A}}_{\mathrm{L}}\; p(\nu_0 - \nu) \sum_{KQ} w^{(K)}_{J_u J_\ell}\; \mathcal{T}^K_Q(i, \vec{\Omega})\; S^K_Q(\vec{x}\,)$$

$$\left(\begin{array}{l} w^{(K)}_{J_u J_\ell} = (-1)^{1+J_\ell+J_u} \sqrt{3(2J_u+1)} \begin{Bmatrix} 1 & 1 & K \\ J_u & J_u & J_\ell \end{Bmatrix} \\[2em] S^K_Q(\vec{x}\,) = \dfrac{2h\nu_0^3}{c^2} \sqrt{\dfrac{2J_\ell+1}{2J_u+1}}\; \dfrac{\left[\rho^K_Q(\alpha_u J_u)\right]_{\vec{x}}}{\left[\rho^0_0(\alpha_\ell J_\ell)\right]_{\vec{x}}} \end{array} \right)$$

- RTE becomes uncoupled four equations

$$\frac{\mathrm{d}}{\mathrm{d}s} I_i(s) = \eta_I^{\mathrm{A}}(\vec{x}) I_i(s) + \varepsilon_i(\vec{x}) \quad (i = 0, \ldots, 3)$$

$$\eta_I^{\mathrm{A}}(\vec{x}) = k_{\mathrm{L}}^{\mathrm{A}}(\vec{x})\, p(\nu_0 - \nu)$$

$$\varepsilon_i(\vec{x}) = k_{\mathrm{L}}^{\mathrm{A}}(\vec{x})\, p(\nu_0 - \nu) \sum_{KQ} w_{J_u J_\ell}^{(K)} \mathcal{T}_Q^K(i, \vec{\Omega}) S_Q^K(\vec{x})$$

- they are readily solved as

internal

$$I_i(\nu, \vec{\Omega}) = \int_{\vec{x}_0}^{\vec{x}} p(\nu_0 - \nu)\ k_{\mathrm{L}}^{\mathrm{A}}(\vec{x}\,')\ \mathrm{e}^{-\tau_\nu(\vec{x}, \vec{x}\,')} \sum_{KQ} w_{J_u J_\ell}^{(K)}\ \mathcal{T}_Q^K(i, \vec{\Omega})\ S_Q^K(\vec{x}\,')\, \mathrm{d}s'$$

$$+\ \mathrm{e}^{-\tau_\nu(\vec{x}, \vec{x}_0)}\ I_i^{(\mathrm{b})}(\nu, \vec{\Omega})\ ,$$

external

where $\tau_\nu(\vec{x}, \vec{x}\,') = \int_{\vec{x}\,'}^{\vec{x}} p(\nu_0 - \nu)\ k_{\mathrm{L}}^{\mathrm{A}}(\vec{x}\,'')\, \mathrm{d}s''$

$$\bar{J}^K_Q(\nu_0) = \left[\bar{J}^K_Q(\nu_0)\right]_{\rm I} + \left[\bar{J}^K_Q(\nu_0)\right]_{\rm E}$$

$$\left[\bar{J}^K_Q(\nu_0)\right]_{\rm I} = \int_{-\infty}^{\infty} {\rm d}\nu \;\; p(\nu_0 - \nu) \oint \frac{{\rm d}\Omega}{4\pi} \sum_{i=0}^{3} \mathcal{T}^K_Q(i,\vec{\Omega}) \int_{\vec{x}_0}^{\vec{x}} {\rm d}s' \; p(\nu_0 - \nu)$$

$$\times \, k^{\rm A}_{\rm L}(\vec{x}\,') \;\; {\rm e}^{-\tau_\nu(\vec{x},\vec{x}\,')} \sum_{K'Q'} w^{(K')}_{J_u J_\ell} \, \mathcal{T}^{K'}_{Q'}(i,\vec{\Omega}) \; S^{K'}_{Q'}(\vec{x}\,')$$

$$= \int_{-\infty}^{\infty} {\rm d}\nu \;\; [p(\nu_0 - \nu)]^2 \int {\rm d}^3\vec{x}\,' \, \frac{k^{\rm A}_{\rm L}(\vec{x}\,') \;\; {\rm e}^{-\tau_\nu(\vec{x},\vec{x}\,')}}{4\pi(\vec{x} - \vec{x}\,')^2}$$

$$\times \sum_{i=0}^{3} \mathcal{T}^K_Q(i,\vec{\Omega}) \sum_{K'Q'} w^{(K')}_{J_u J_\ell} \, \mathcal{T}^{K'}_{Q'}(i,\vec{\Omega}) \; S^{K'}_{Q'}(\vec{x}\,')$$

$${\rm d}^3\vec{x}\,' = (\vec{x} - \vec{x}\,')^2 \; {\rm d}\Omega \, {\rm d}s'$$

$$\left[\bar{J}^K_Q(\nu_0)\right]_{\rm E} = \int_{-\infty}^{\infty} {\rm d}\nu \;\; p(\nu_0 - \nu) \oint \frac{{\rm d}\Omega}{4\pi} \sum_{i=0}^{3} \mathcal{T}^K_Q(i,\vec{\Omega}) \;\; {\rm e}^{-\tau_\nu(\vec{x},\vec{x}_0)} \; I^{\rm (b)}_i(\nu,\vec{\Omega})$$

$$\left[1+\epsilon+\delta_u^{(K)}\right] S_Q^K(\vec{x}\,) + \mathrm{i}H_u \sum_{Q'} \mathcal{K}_{QQ'}^K\, S_{Q'}^K(\vec{x}\,) =$$

$$= \delta_{K0}\,\delta_{Q0}\,\epsilon\, B_{\mathrm{P}}(T) + w_{J_u J_\ell}^{(K)}\,(-1)^Q \left[\bar{J}_{-Q}^K(\nu_0)\right]_{\mathrm{E}}$$

$$+ \int \mathrm{d}^3\vec{x}\,' \; \frac{k_{\mathrm{L}}^{\mathrm{A}}(\vec{x}\,')}{4\pi(\vec{x}-\vec{x}\,')^2} \sum_{K'Q'} \underline{G_{KQ,K'Q'}(\vec{x},\vec{x}\,')}\; S_{Q'}^{K'}(\vec{x}\,')$$

where

$$\underline{G_{KQ,K'Q'}(\vec{x},\vec{x}\,')} = \int_{-\infty}^{\infty} \mathrm{d}\nu\, \left[p(\nu_0-\nu)\right]^2\, \mathrm{e}^{-\tau_\nu(\vec{x},\vec{x}\,')}$$

multipole coupling coefficient

$$\times\, w_{J_u J_\ell}^{(K)}\, w_{J_u J_\ell}^{(K')} \sum_{i=0}^{3} (-1)^Q\, \mathcal{T}_{-Q}^K(i,\vec{\Omega})\, \mathcal{T}_{Q'}^{K'}(i,\vec{\Omega})$$

- equation for plane-parallel, semi-infinite stellar atmosphere is obtained as

$$\left[1 + \epsilon + \delta_u^{(K)}\right] S_Q^K(t_{\mathrm{L}}) + \mathrm{i} H_u \sum_{Q'} \mathcal{K}_{QQ'}^K \, S_{Q'}^K(t_{\mathrm{L}}) =$$

$$= \delta_{K0}\, \delta_{Q0}\, \epsilon\, B_{\mathrm{P}}(T) + \sum_{K'=0,2} \int_0^\infty \mathcal{G}_{KQ,K'Q}\big(|t'_{\mathrm{L}} - t_{\mathrm{L}}|\big)\, S_Q^{K'}(t'_{\mathrm{L}})\, \mathrm{d}t'_{\mathrm{L}}$$

where $$\mathcal{G}_{KQ,K'Q}\big(|t'_{\mathrm{L}} - t_{\mathrm{L}}|\big) = \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}y' \, \frac{\Delta\nu_{\mathrm{D}}}{4\pi(\vec{x} - \vec{x}\,')^2} \, G_{KQ,K'Q}(\vec{x}, \vec{x}\,')$$

$$G_{KQ,K'Q'}(\vec{x}, \vec{x}\,') = \int_{-\infty}^{\infty} \mathrm{d}\nu \, \big[p(\nu_0 - \nu)\big]^2 \, \mathrm{e}^{-\tau_\nu(\vec{x}, \vec{x}\,')}$$

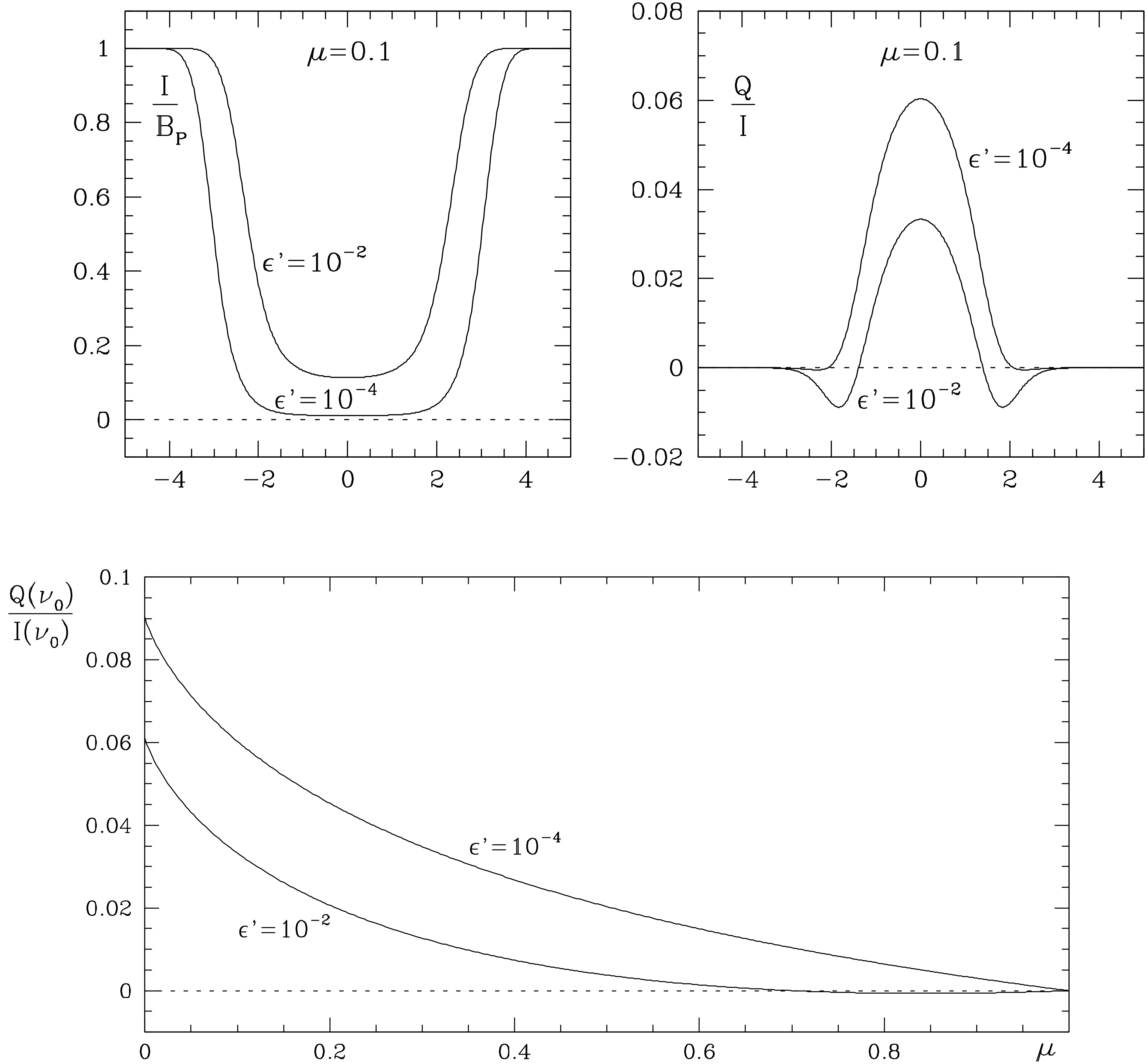
$$\times\, w_{J_u J_\ell}^{(K)} \, w_{J_u J_\ell}^{(K')} \sum_{i=0}^{3} (-1)^Q \, \mathcal{T}_{-Q}^K(i, \vec{\Omega})\, \mathcal{T}_{Q'}^{K'}(i, \vec{\Omega})$$

- Stokes parameters $I_i(\nu, \vec{\Omega})$ are derived from S^K_Q as

$$I_i(\nu, \vec{\Omega}) = \int\limits_0^\infty \varphi(v)\ \mathrm{e}^{-\frac{t_\mathrm{L}\ \varphi(v)}{\mu}} \sum_{K=0,2} \sum_Q w^{(K)}_{J_u J_\ell}\ \mathcal{T}^K_Q(i, \vec{\Omega})\ S^K_Q(t_\mathrm{L})\ \frac{\mathrm{d}t_\mathrm{L}}{\mu}$$

or

$$I_i(\nu, \vec{\Omega}) = \sum_{K=0,2} \sum_Q w^{(K)}_{J_u J_\ell}\ \mathcal{T}^K_Q(i, \vec{\Omega}) \int\limits_0^\infty \mathrm{e}^{-\tau_\nu}\ S^K_Q\left(\frac{\mu\, \tau_\nu}{\varphi(v)}\right) \mathrm{d}\tau_\nu$$


μ=0.1
I/B_P
ε'=10^-2
ε'=10^-4
μ=0.1
Q/I
ε'=10^-4
ε'=10^-2
Q(ν_0)/I(ν_0)
ε'=10^-4
ε'=10^-2
μ